**Thermal Properties**

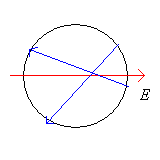
**The Free Energy (Functional)**

So recall in the last file, we more or less derived the Ginzburg-Landau free energy for a superconductor along with the magnetic field it’s in (including its own contributions to that field)



where F = Fs + Fn, and F is the total free energy of the superconductor, and Fn the free energy of the superconductor if we were to turn off the attractive interaction, or in other words, if we were to set Δ = 0 [for Δ, see Excitations folder, and in the context of the BCS Hamiltonian, setting Δ = 0 would just be the free fermion model which we’ve studied in the Stat Mech folder, and Metals/Free Day folder]. This is like how in the Excitations file we found it useful to split calculations of certain Δ-dependent things, X, into X(Δ) = [X(Δ) – X(0)] + X(0) = Xs + Xn.

Back to ψ(r). So by analogy with spins, we can at least say that it is the ‘complex’ order parameter. In the ferromagnet case, φ(r) could ultimately be interpreted as basically the magnetization, which served as the order parameter too. And here we would ask what will ultimately be the interpretation of this ψ. And I believe it turns out to be proportional to Δ(r,T), the inverse Fourier transform of Δ(k,T), where Δ(k,T) = Δ(T)θ(|ξk| < ωD) ~ Δ√(1-T/Tc)θ(|ξk| < ωD). This is reasonable because we saw that Δ(T) had the same critical temperature properties as the order parameter in the spin problem and so it makes sense as an order parameter. Now that we have an EM field in our problem, we should probably update Δ(r,T) → Δ(r,T,**A**), which will likely be a little different than just Δ(r,T). Whatever. It will also turn out to be proportional to the Cooper pair probability density *amplitude* (|ψ|2 is proportional to probability density for eligible particles – those within ωD of Fermi surface – to form pairs). We’ll see this interpretation play out in a bit. And last, we can say that it is proportional to the probability density amplitude that an electron is participating in the super current, so something like (well, exactly equal to in fact) √ns\* where ns\* is the density of electrons (or electron pairs) participating in the supercurrent. Note that ns can range from no electrons participating, to all of them. Even if an electron is not within the Debye wavelength range of the Fermi surface, where the attractive interaction is present, and so thereby cannot form a Cooper pair per se´, it can still be part of the supercurrent because it isn’t scattering (recall that picture of how the whole Fermi ‘sphere’ accelerates under the influence of the electric field, but only the surface electrons scatter – or not in the superconducting case)



So to summarize, suppresing T, A arguments…



**GL Free Energy and Equations of State**

Now we need to get the equilibrium Fs by minimizing w/r to the internal ψ(x) d.o.f. So taking functional derivative and all,



Taking the complex conjugate of this equation, and calling the solution ψeq, we have:



(the square of that operator is like dot product square kind of thing) So this implicitly fixes ψ(x) in terms of T and **A**(x), and gives us our equilibrium Free energy. And it implicitly fixes the probability density of Cooper pairs in our metal too, again, as a function of T and **A**(x). So now we have:



[Gaussian units] Next we need the differential relations obeyed by the independent variables of the free energy. We will still have as usual the one for entropy.



where F = Fn + Fs. And as for the field **A**(x)…well recall that the first law for a system that includes the external field is (see Thermodynamics/Equilibrium Systems and/or EM/Insulator Energy)



δWmech just refers to pressure work and the like. Going to neglect it. The penultimate term is the differential work we do to create the (total) magnetic field in our superconductor (or insulator, whatever) - note Bf is just μ0H → 4πH (in faux Gaussian units), at least for a solenoidal geometry.



Want to put this in terms of the vector potential. So,



So our 1st law would read, for equilibrium transformations of the entropy and vector potential:



And so the differential relation for the Free energy F = E – TS would read:



And so in particular,



and of course jf is the ‘free’ current density, as opposed to the ‘bound’ or ‘induced’ current density. Now let’s work this out and see what we get. First F = Fn + Fs, and presumably Fn has no **A** dependence. So just have to worry about Fs. Going to call Fs1 the part of Fs with ψ in it, and Fs2 the B2 part. Playing loose with fact that **A** is a vector.



where = -i∇, is the canonical velocity operator. And for the other guy, using **B** = ∇×**A**, and putting in terms of Levi-Cevita symbol.



There’s that. Think we’re sticking with the Coulomb gauge, so we can say this goes to:



And in said gauge, Maxwell’s equation says ∇2A = -(4π)j, so:



So altogether,



And equating this to **j**f, from our highlighted Thermodynamic relation, we see that.



Since j = jf + jb, where jb is the bound current, i.e., the current coming from the superconductor, we obviously have:



which can be written in terms of the physical velocity operator:



The exact analogy between this expression for the bound current and the quantum mechanical expression for the current within a wavefunction ψ(x) [see QM/Scattering I guess] substantiates our provisional interpretation of ψ as a sort of macroscopic wavefunction/probability amplitude for the location of the Cooper pairs/supercurrent electron density. And now we have an equation for the current in our metal. With the free energy specified thus, we can determine the thermodynamic properties of our material, at least near the critical point.

Might be worthwhile to work this out a little. Let ψ(x) = √ns\*eiφ(x). I’ll say ns is constant for now. Then,



and this matches the form we derived in the classical Meisner effect analysis,



**Probability Density/Current Density of Cooper Pairs in absence of field**

Let’s take a look at the equilibrium value of ψ. ψ(x) is governed by:



(again square is in dot product operator) sense. Let’s look at the case where we don’t have a field. So A = 0. And try a solution of the form ψ = √(ns\*)eiφ(x). Then we have:



So when r < 0 we get spontaneous creation of Cooper pairs/super current. And r of course is just proportional to T – Tc, whatever Tc is, and whatever the proportionality is [these things not specified by the theory]. For what it’s worth, we’ll note that the phase of ψ(x) is completely unspecified. And yet, the system will have to assume *some* phase when it condenses into the superconducting state. This is an example of broken symmetry, just like how the phase of the magnetization in the Ising spin problem could be either positive or negative. One interesting difference is that unlike in the magnetization problem, where this broken symmetry gave rise to Goldstone excitation modes (i.e., modes – in that context called spin waves – which had excitation energy arbitrarily close to the ground state and hence no ‘mass gap’), we don’t have the same here because excitations have to surmount the 2Δ gap. There’s a reason for this but don’t know. From above, the current corresponding to this would be:



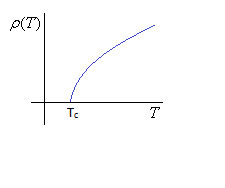
which probably makes sense, as the lowest energy state would be w/o current. What if we don’t assume ∇ψ = 0? Let’s try a plane wave solution ψ(x) = √(ns\*)eikx. Then we have:



Not sure how to interpret this. First, in order for ns\* to be positive, we need k2/2m\* + r/2 < 0. So T must be less than Tc by such an amount to compensate for the positivity of k2/2m. And it seems that giving them all this phase militates against forming Cooper pairs. Let’s consider the current density under the same conditions:



So seems this solution (and I guess any solution with phase) would have a non-zero current in thermodynamic equilibrium, and moreover, in the absence of any EM field. This would be the super-current. And we know it would theoretically last forever because, as it’s in equilibrium, it’s not generating any entropy. But from the previous result it seems that the greater k is, the smaller ρ would be, and so seems that there is an upper bound to the supportable current? Presumably, if we were to calculate the resistivity, we’d find something like this, i.e., 0 resistance for T < Tc, and then finite afterwards.



**Length Scales**

It will behoove us to consider a physical interpretation of some of the terms in Fs. So recall,



Maybe just consider the terms intrinsic to the superconductor (the non-A ones). So generically, we’ll observe that if r < 0, then the superconductor favors creation of Cooper pairs, i.e., non-zero ψ. But even still there is a limit because as ψ gets larger, the ψ4 term ensures that it begins to cost energy to create such pairs. And the ∇ term implies there is an energy cost to ψ having spatial fluctuations.

Going back to that A term, we can see that there is an energy cost to having a magnetic field in the superconductor. From the Meisner effect, we can see that non-zero ψ can expel A and so for r < 0, it would seem that creating Cooper pairs would definitely be favored so as to eliminate A. But there is a limit because the ψ required to expel very large A’s would eventually cost energy thanks to the ψ4 term. Apparently another way to eliminate (some) A is to create spatially varying ψ. This costs energy too, thanks to the ∇ term, but I guess not as much as having constant ψ does via the ψ4 term. Type II superconductors seem to favor this approach to expelling (some) A.

Well anyway, looking at the intrinsic terms again, it seems there are two length scales.



Comparing first two terms, we can see that ∇2/m\* and r must have the same units. ∇ has units of 1/length. Therefore 1/√(m\*r) must have units of length. So one customarily defines a ‘correlation length’,



This length is pertinent to Type II superconductors. It is the radial length scale over which a flux quantum penetrating the superconductor manages to eliminate the Meisner current which would otherwise expel the A field (I guess it’s the size of the vortex). If ξ is large then the superconductor is considered ‘stiff’, and ‘lose’ otherwise. This is in analogy with the fact that if we placed a mass on a stiff rope, then this would induce a large length of the rope surrounding the mass to cave downward. But if the rope were loose, then only a small neighborhood around the rope would do so. If we factor ξ out of [Fs]intrinsic, we’d have:



And so we can see that the stiffer around superconductor is, the greater the energy cost in having fluctuations, ∇ψ. For comparison’s sake, I’ll just recall the other length scale we ran into – the penetration depth:



which is the distance to which the currents will persist below the surface of the superconductor, and recall ns\* = -r/u.