**Inverse Participation Ratio**

Turns out the spatial structure of the wavefunction at the critical point is quite interesting. A delocalized wavefunction spreads over the entire sample and assumes the sample’s dimensionality. A localized wavefunction has an effective dimension of 0. But the critical wavefunction has a fractal dimensional aspect. We can mathematically define the radius of the wavefunction, R in terms of the wavefunction as follows. We define the inverse-participation ratio P-1 as:



where the integral extends over the entire region (perhaps length or dimensions of the wire). Note that P-1 has units of Ld∙(L-d/2)4 = L-d → P ~ Ld. So P can be considered to be the average volume of the state, and P1/d may be considered its average radius, R. Note that Ld isn’t necessarily the volume of the state if it is localized *within* that volume (perhaps we could do a similar thing for gasses in thermodynamics for purposes of defining their volume – just switching out ψ for ρ).

**Effective dimensionality**

Suppose P ~ Ld\*. This d\* is called the fractal dimension of the state. We may imagine d\* < d, and the tinier it is, the more localized-like it is. The state will be completely localized when d\* = 0. For instance, if the state were localized, then |ψ|4 would not reach the ends of the sample, and so P-1 would not depend on the size of the sample, and so it P-1 would just give an L-independent constant. So when d\* = 0, then we’d just have P = Rd. One may consider higher moments and introduce correspondingly higher fractal dimensions. If these higher d\*’s are not integer multiples of d\*, then the wavefunction is considered to have *multi-fractal* behavior.

**Ex. delocalized state**

For instance consider a constant state ψ = A. The normalization would require A to be like A2Ld ~ 1 → A ~ L-d/2. And so then,



So the volume of a constant state is just that of its container, which makes sense. What about a state that decays as A/rq. Then normalization requires:



And P would be:



This is pretty typical, in that we do not consider a power law envelope to be characteristic of a *localized* state.

**Ex. localized state**

What about an exponential decaying state, ψ ~ Ae-r/ξ. Then normalization requires:



And so



And this is expected too. So an extended state has P ~ Ld, and a localized state goes as L0. We can characterize the degree of localization by calculating P in the asymptotic large L limit.