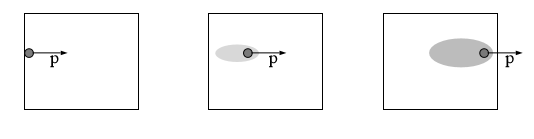
**Interaction**

**Forward**

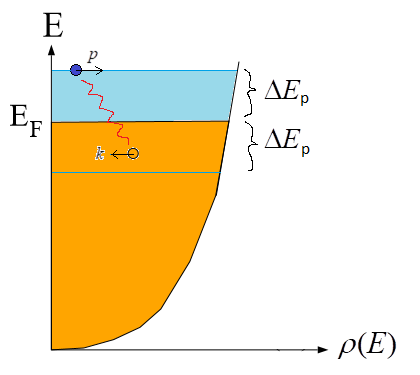
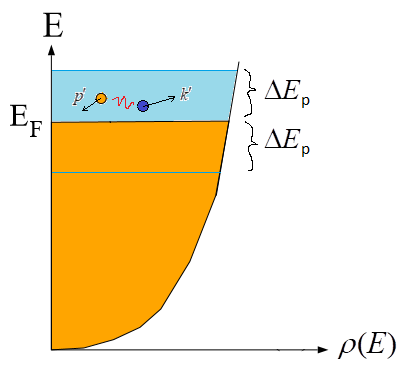
These notes were originally written for liquid 3He (see Quantum Fluids). But I modified them to include in this file. Might observe that Fermi liquid theory can be substantially derived by applying a Renormalization Group analysis (see Stat Mech folder, or Exchange Interaction Folder) to the bare e-e interaction Hamiltonian (see Nearly Free electrons folder). I haven’t done this, but I imagine it’d go something like this. We form the path integral expression for the (grand) partition function. Then separate the ψk into short wavelength k’s (Λ/b, Λ) and long wavelength k’s (0, Λ/b). Integrate out the short k’s to get an effective action for the long ψk’s. This will give us an effective Hamiltonian for the low lying excitations of the system (small k’s are lowest lying excitations). The effective Hamiltonian will actually contain many terms, but I’m guessing that just like with ferromagnet analysis in Stat Mech folder or Exchange Interaction folder, some of the terms will just renormalize the free mass and Fermi velocity: m(b) = bλ\_mm, vF(b) = bλ\_evF (temperature is a prefactor for everything, but can consider temperature fixed I think – just a matter of point of view) as a function of scale, b. And, continuing the analogy, I think other terms will account for additional interactions and go as bλ\_int. Seems that the only interaction term that won’t scale to zero in the large b limit (i.e., have negative eigenvalues λ\_int) is the δnpσδnp´σ´ term (see below). Also seems that this interaction term will have four independent parts corresponding to the F0,1a(b) and F0,1s(b) parameters. And moreover, it looks like the RG equations for m(b), vF(b), F0,1a(b), F0,1s(b) will have a fixed point at some m\*, vF\*, F0,1a, F0,1s which correspond to the Fermi liquid parameters discussed below. This fixed point will correspond to T = 0. And so keeping just the terms in the effective action that don’t scale to zero in the large b limit, and using the fixed point coupling constant values, we have an effective action/Hamiltonian for the long wavelength modes of an interacting electron gas. The Fermi liquid theory presented below is a sort of phenomenological account of this theory.

**Introduction**

Fermi Liquid Theory is a phenomenological approach to treating the e-e interaction. FLT does not suppose the interaction to be weak. It *does* suppose that we can treat the elementary excitations of an interacting e gas as quasi-particles. That the excitations of an interacting e gas take the form of quasi-particles is familiar perhaps from our perturbative analysis of the nearly free interacting electron gas; of course that analysis was just perturbative. In this model, we (will) explicitly find that, at least near the Fermi surface, quasi-particle excitations have a really long lifetime that goes as τk ~ (EF/(Ek-EF))2. Might venture a qualitative calculation along those lines now. Consider injecting an electron with momentum p > kF into the ‘liquid’. As soon as we do,



we’ll note that momentum injected into the system will be conserved, but the interactions between it and the other electrons will distribute this momentum amongst the other electrons. *Eventually*, the interactions will slow the electron down to where it drops to somewhere just above the former Fermi surface. But if the interactions are few, and if the injected electrons is close to the original Fermi surface, it will keep most of the momentum for a long time, and so we can think of this particle-like excited state to be close to a legitimate eigenstate of the system, (as true eigenstates are infinitely lived). Being injected close to the Fermi surface is the predominate contribution to the long lifetime though. For consider an electron injected with energy ΔEp above EF (red squiggle is interaction), and scattering with another one within the Fermi sphere.

If the Pauli-Exclusion principle didn’t apply, then it could scatter with any other electron within the Fermi sphere. I suppose it technically could even still, if it were to happen that the scattering event caused the two electrons to precisely exchange places. But this is astronomically unlikely (well at least it would be in 2D or 3D – but in 1D I guess there’s a 50/50 chance? which is perhaps why 1D electrons don’t act as Fermi liquids?) So for practical purposes, the two must scatter into a space above the Fermi surface. And they can only do that if the other electron is within the same energy window ΔEp below EF as the injected electron is above it. We can be somewhat quantitative. The inverse lifetime of our injected particle would be, approximately [see Condensed Matter/Metals/e-e interaction/Nonequilibrium/Conductivity file for more on calculating this inverse scattering time]:



What is the matrix element? Well,



switch to c.o.m. variables,



Jacobian is 1.



Now let’s consider the potential Fourier transform. Since we’re dealing with electrons, which would be screened, thanks to the other electrons roaming around and the positive ionic background, we could use the simplest screened model – the Thomas-Fermi approximation. We got (see Condensed Matter/Metals/Electron-Electron/Thermal Equilibrium properties folder):



But this is the potential, not potential energy, so we have:



But could be even lazier, and just use the q = 0 result. This would put us at least in the ball park, and is equivalent to using a delta function approximation to the potential. So we’ll go with that approximation,



Continuing,



Doing the sum over k will give us:



And the sum over k´, p´, would be restricted to area above Fermi surface thanks to 1-nF(εk´) and 1-nF(εp´) factors. So,



The nF left seems to restrict the p´, k´ sums to within that aforementioned energy window ΔEp above the Fermi surface now. So



where the Δp reminds us that the radial part of the k´, p´ integration only proceeds within that momentum window (energy window). Gonna rough it from here on out. I’ll change coordinates from **p** → εp, Ω, and write:



Now if the energy window is small, we can just approximate the magnitudes k´, p´, and p, as being kF. And then we have:



where in the last we do the radial integration, and am implicitly measuring angles w/r to the direction of **p**. Using ρ(εF) = 3n/2εF, we have:



The angular integration is just some number of order 1. And so just have:



And can see that as ΔEp → 0, so does the scattering rate. Found a dude’s nice illustration of the spectral function of single particle GF, as we move k further from the Fermi Surface.

Diagram

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So can see that the excitations do strongly mimic the form of a free-particle excitation with a renormalized energy. That an interacting gas behaves as a free gas, in some sense, also finds support in the fact that things like the specific heat, etc., of strongly interacting gasses looks just like that of free gasses (sans a few renormalized parameters), at least at low T.

**Interacting States**

So to make this correspondence between the many-body free states and the many-body interacting states more precise, Landau’s idea was as follows. We start with a free gas – in its ground state or various low energy eigenstates – and propose adiabatically turning on the interaction to get to the interacting gas. The adiabatic theorem guarantees that there is a one to one correspondance between the free and interacting states (if they’re non-degenerate at least) as long as there are no energy level crossings, as a function of V. This will not be the case for instance, for superconducting metals. For there we start with, in the non-interacting case, a bunch of uncorrelated electrons. But in the fully interacting case, we don’t have the ground state being a bunch of uncorrelated electrons. Instead, we have a certain spin (Cooper) pairing of electrons which lowers the ground state energy below the energy of a bunch of uncorrelated, unpaired, but interacting, electrons. Of course these features are mediated by the e-ph interaction, and we don’t have that here. Anyway, we’ll presume not to run into these problems.

So consider the free gas in its ground state or any excited state close to its ground state. We can write it as |Ψ{n\_pσ}>0, where npσ is a set of numbers that gives the # of particles that have momentum p and spin σ [doesn’t have to be just a single particle excitation]. Then we imagine slowly turning on the interaction between the particles. This will turn it into the state |Ψ{n\_pσ}>. Note that in the presence of the interaction, the numbers npσ no longer necessarily have the meaning that npσ particles have momentum p and spin σ; the interactions can/will change each individual particles momentum (well, not the spin, but definitely the momentum). So npσ only means that we’re in the interacting state that *corresponds* to the free state with those occupation numbers. We could use elementary perturbation theory to see what this state would look like, to first order in the potential interaction. But for a strong interaction, Feenburg and his associates have determined a good ansatz for these wavefunctions |Ψ{n\_pσ}>. He says, to good approximation, they just look like this:



where |Ψ{n\_pσ}>0 is just the Slater determinant of free wavefunctions/spinors and Φ is a highly correlated many-body state, similar to the one used to describe 4He. So,



and:



where ψpσs(r) = (1/√V)eip·rχσs, and χσ is some spinor, and the s refers to its components along, say, the z direction. Typically χσ would be spin up or down in the z—direction. And so χσ=+s = (1 0) and χσ=-s = (0 1). But if χσ were spin up/down in the y or x direction, then the components would be different of course. Again, one would expect that this ansatz would only work for the GS and ES close to it. I wouldn’t expect that starting with just a couple particles in random free-particle states would retain much of there free particle character even if we did slowly turn on the interaction, because there would be many states they could scatter in to. Can we work out the normalization of this state?



Yeah that looks ugly. But we’ll normalize our wavefunction so that it gives us 1 somehow.

**Interacting States’ Momentum and Spin**

Now consider the average momentum. We can calculate it in principle…



So we have:



And this result was the momentum of the original unperturbed state. So these states preserve total momentum. But note that the momentum of each individual particle is not the same as it was before, or even a constant. *Now* the particles are constantly interacting and exchanging momentum. But total momentum is conserved, as it should be, since V is a pair-wise force. Spin is likewise conserved, since the interaction is spin-independent. We can make further conclusions. For instance we may ascertain the quasi-particles are spin ½ fermions just like they were before the interaction, because thanks to the one-to-one adiabatic correspondance, npσ can only be as much as one in the interacting case, since the same holds true in the free case. Unintuitively, I read that even for spin s > ½ fermions, the quasi-particles themselves must still be s = ½. Now even though the interacting state is a true many-body state, I think we can still think of it as something very like the same collection of particles whereby npσ of them have momentum p and spin σ. This is because as we turn on the interaction, the electrons can hardly scatter out of these states (as long as we’re close to the ground state) thanks to the Pauli Exclusion principle. We can calculate the momentum distribution function: nk = Σσ<ckσ†ckσ>. Recalling the Many-Body Correlations file in the QM folder, we first form the N-particle correlation function:



(I’m assuming we’re in Ground State, say). And then integrate/sum over all coordinates/spins, except for the first one, to get the single particle correlation function:



and then take the Fourier transform w/r to **r**-**r**´, and take the Trace over spins.



One finds:

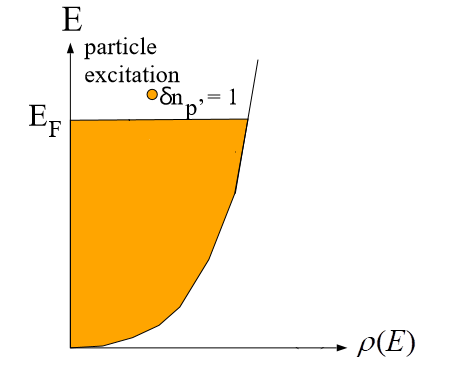
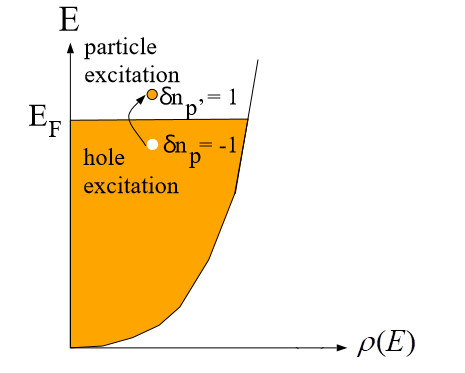
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So evidently the interaction gives some of the electrons a higher kinetic energy than they had before, at least some of the time (since they don’t continually occupy a single momentum state). But can also see that the position of the Fermi surface k = kF is preserved by the interaction. I think there is a proof that interactions will always preserve the location of the Fermi surface like this. This is Luttinger’s theorem?

**Interacting States’ Energy**

Just as we calculated the momentum expectation of a given state, we could go ahead and calculate the energy expectation too. I think Feenburg does this calculation but we’re not going to. We’ll just follow Landau’s arguments. So since we only need the low lying states, we’ll write their energies in terms of deviations of occupation numbers from the GS. The GS is given by the configuration npσ = 1 up till p = kF (but remember that npσ doesn’t mean that there are npσ electrons with momentum p and spin σ in the interacting state, but rather there *were* in the free state, before the interaction was turned on; still, as we argued vis a vis the quasi-particle picture, we can pretend there are kind of npσ electrons with momentum p/spin σ in the interacting state). Landau presumed the energy could be written as a functional of δnpσ, where δnpσ is the deviation of the distribution from the ground state (the zero-magnetic field ground state, also called the ‘unpolarized’ ground state), and moreover that we could expand it in powers of δnpσ basically. Two given excitations are shown below. In the first, we simply add a particle to the system. In the second we excite a particle within the system.

So anyway, we might write the energy expectation of a state |Ψ{n\_pσ}> as something like:



where EGS is the energy of the interacting ground state. Note εp(0) is *not* the free particle energy k2/2m. εp(0) is the energy (above ground state) of a single quasi-particle excitation all alone. Since we’ll only be dealing with small deviations from the Fermi surface, we can simply make the linear approximation to the spectrum near the Fermi surface and say:



Might be easier to write it like this:



and this would only hold to linear order. Don’t know what εF\* is (well, it’s the chemical potential at T = 0), though we do know it occurs at the same kF as it does for the non-interacting case. But we shouldn’t need it. m\* would be a phenomenological parameter to determine from experiment. But we don’t stop there. Landau realized that the Σpσεp(0)δnpσ term is actually second order in smallness, as p-pF is presumed small itself. So for self-consistency, we must go out to (δnpσ)2 in our E expansion and include that f term which basically models interactions between quasi-particles. Another way of looking at it: in order for this expansion to be valid, we must have (on average) Σpσδnpσ ~ O(εp(0)) → δnpσ ~ O(ε­p(0)/N). This way the last two terms in E will be comparable. So both the εp(0), and the fppʹσσ’ guy are phenomenological terms that must be determined by experiment basically. Note some symmetries:



It will also turn out that fpσ;p´σ’ is symmetric w/r to interchange of p and p´, as well σ and σ´. One thing of note is that E[npσ] depends on the orientation of the spins. Explicitly separating out terms,



we see that if we flip all spins so that all formerly pointing up are now pointing down, and vice versa, then we get the same energy, as we should. But if we take just a single spin, and flip it, then we get a different energy, thanks to the f-term. This means that the f-terms can take into account the Pauli Exclusion principle – because switching local orientations of spins will push similarly oriented spins away, and thereby alter potential energy. So basically, if we specify all the particles’ spins, then this, in conjunction with the potential, determines how close to each other they get, and therefore their energy.

It is common to make simplifications on the d.o.f. of fpσ;p´σ´ too. First, since our quasi-particle concept only applies to excitations close to the Fermi surface, we restrict p and p´ to this surface. If we do this, then fppʹσσ´ only depends on the angle, θ, between p, pʹ. Now there are four spin combinations – up/down × up/down. In an isotropic Fermi liquid, it should be invariant to rotation. Would expect the like-spin interactions should be same, and unlike should be same. So we expect something like,



ρF\* = m\*kF/π2ℏ3 is there just to factor out the units and make F, G unitless. This is the density of states near the Fermi surface, which follows from the form of the energy spectrum near the Fermi surface) Could equivalently write this as:



where σ, σ´ = ±1. Then we could expand the angular functions in a series of Legendre polynomials. So,



where,

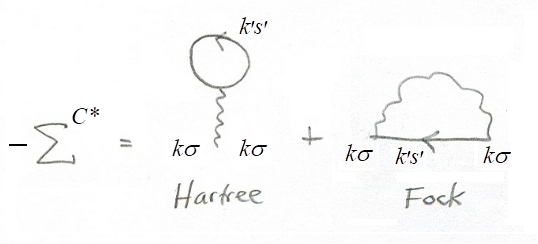


Keeping just the first two terms in the expansion is called the s-p approximation.



**fpσ;p´σ´ to first order in V**

For definiteness, we can work out what f is in the limit of weak interaction. Let’s consider the self-energy of an electron. To first order in the interaction, it’s given by:



Recalling the Feynman rules (maybe see e-e interaction file on single particle excitations), we have, at some finite temperature T:



So altogether,



Now if we have a distribution of quasi-particles npσ, then the energy of a given one, in the presence of all the others, can be obtained via:



And so we have:



Or another way to write it,



And we can write:



And so in the weak interaction limit, we’d have:



Comparing with our general result,



we have,



and so,



**fpσ;p´σ´ constructed phenomonologically from experimental results**

From Lecture Notes on Quantum Matter (Steven Simon), I found these typical values (well these are actually for liquid 3He):

A picture containing table

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Interesting that pressure plays such a large role, but I guess that affects the interatomic spacing, and therefore the potential they all feel? Anyway, so at 1atm, this means the effective interaction term would be:



So there. Might note that the second term is evocative of the exchange interaction between two electrons in nearby orbitals (see Quantum Mechanics/Many Identical Particles),



In our case, J would be slightly positive, and so this would be a ferromagnetic exchange interaction.

**Relating ρF\* to ρF, i.e., m\* to m**

So we have 5 d.o.f. – m\*, F0a,s, F1a,s. And turns out we can relate m\* to F1s, which reduces us to 4 d.o.f. Relationship seems to have to do with the fact that the quantum number p still identifies momentum of a quasi-particle, but also that the group velocity of the quasi-particles is given by ∇pεp [remember group velocity of a plane wave superposition is ∂ω/∂k]. And so its momentum could be said to be m∇pεp [this could use some justification sometime]. And these two must equal. Or more generally, must have:



(but note we’ve restricted p to Fermi surface, really, so maybe this should be just a surface integral, maybe times a shell thickness ∫dΩ/4π·ρ\*F(ε)dε) We’ll compare these guys out to first order in δn. Don’t need to go higher?



To zeroth order we have:



since we’re integrating an isotropic function. Next, the first order term is:



and in the second line we interchange pσ, p´σ´ labels. Now we can equate the coefficients of the first order terms (remember **p** is implicitly restricted to Fermi surface though)



where we integrate by parts. Now we have to be more careful about taking derivative of npσ(0). We know npσ(0) = θ(εpσ < εF), but how does εpσ depends on p? Well we have:



where the middle line follows from the fact that we’re in the ground state so that δnpσ = 0. So then we can say,



Below in the d3p´ integral, let’s align our p´z axis with **p**. So then,



I guess we can fill our f in,



Now the σσ´ should go away because its odd in σ´ and so sum will cancel it. The F0s term will go away because integrating **p**´ around a circle ought to give 0. And then we’ll be left with the F1s term. We can see that the integral will come out in the direction of **p**, since cosθ´ is angle between **p**´ and **p**. So we’ll just take the **p** component of things. This will give us:



Filling in the density of states:



and so,



It seems that F1 is usually positive, which would make m\*>m, which would make sense if the quasi-particles are dragging a quasi-particle cloud with themselves. And so this also makes the density of states,

