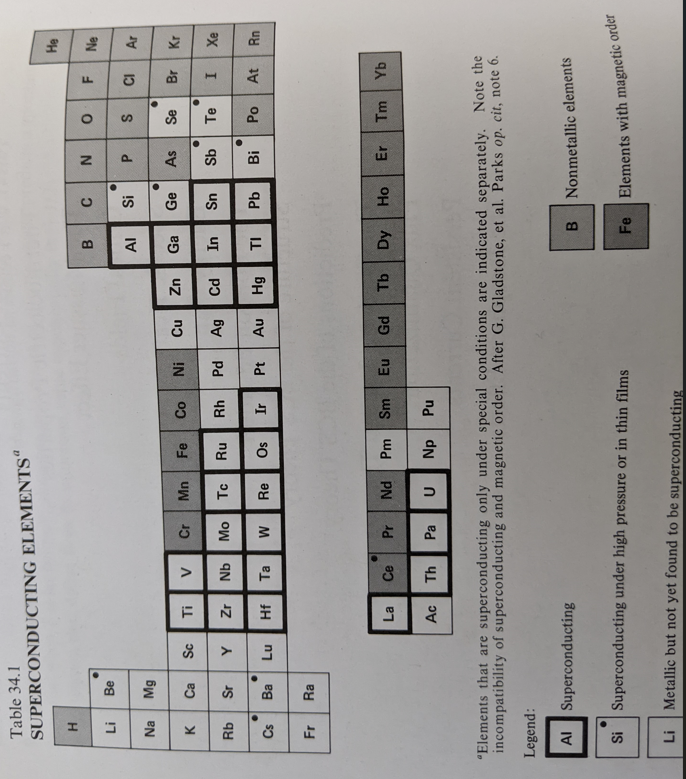
**Superconductivity Interaction**

**Introduction**

Superconductivity does not occur for all metals. Consider following table from Ashcroft/Mermin.



The metal must have a strong enough e-ph interaction to induce a sufficient attractive force between e’s (this is why superconductors are often poor conductors at room temperature – the strong e-ph interaction will result in a lot of electron scattering). And the metal must have a large enough electron density of states. Or well, the critical temperature below which these metals would superconduct depends on these parameters. Basically Tc ~ exp(-1/λρF). So if the electron-phonon coupling is weak, then λ will be small. And if density of states is small, then ρF is small obviously. And this would make Tc super small, perhaps beyond presently achievable temperatures. Additionally, the metal has to support its electrons joining up in ST = 0, i.e., ↑↓ singlet pairs. So Alkali/Alkaline metals don’t superconduct (at achievable temperatures anyway) because their density of states is too small and or their electron- phonon coupling is too weak. But the transition metals often do superconduct, as often their Fermi surfaces intersect the d-bands and so their density of states is large; the Noble metals are an exception. But the transition metals that have a ferromagnetic phase transition at low T’s will not superconduct, because their spins would tend to want to pair the wrong way, i.e., ↑↑ or ↓↓. Aℓ does have a barely strong enough coupling strength and density of states to superconduct. Some metals, like Pb, couple very strongly, and require the Eliashberg theory for an adequate description.

So if consider a superconductor, and postulate its GS to be that of a normal metal, we’ll find it to be unstable – meaning that the isothermal compressibility is negative or something I suppose – which indicates that the metal prefers to be in another state (a SC one as it turns out). These bound states begin forming at TC, and are between the electrons within a Debye energy of the Fermi surface, since that’s the region of phase space where the potential is slighly attractive. As T → 0, more and more of these eligible electrons form bound pairs. And at T = 0, all of them are bound. In parallel, as T goes from Tc → 0, more and more of the electrons are superconducting. At T = 0, all of them are (not just the ones forming bound states). All of them are superconducting because none of them can scatter out of their Fermi surface ‘volume’, thanks to the energy gap that develops, cutting that volume of phase space off from the rest of the excitations spectrum. This energy gap, 2Δk(T), grows from 0 at Tc to, well, 2Δ, at T = 0.

The bound states of the electron pairs are not described by simple orbitals such as used for the hydrogen atom or positronium. To elaborate, basically pairing occurs between eligible electrons in states with opposite momentum and spins, (k, σ) and (-k, -σ). We might suppose this to be because such particles have the same energy and so would be more susceptible to the binding potential since they occupy a degenerate subspace. The two spins are combined in a spin singlet (S = 0). The singlet was chosen in BCS theory on the basis that the other choices of spin combination lead to a triplet state S = 1 which would imply paramagnetic properties in the SC state, which are in fact absent. Later work by Balian and Werthamer (1963) showed that the triplet state had smaller binding energy and was therefore less favored anyway. However, recent theories of 3He superfluidity are based on the premise that the pairing occurs in the triplet state.

The bound state energy gap, Δk(T), in metals should reflect the symmetry of the crystal. Electron states in crystals belong to representations of the group which leave the crystal invariant. Angular momentum eigenstates, such as s-, p-, etc., do not belong to representations of crystal groups. Instead, the crystal representations are *mixtures* of angular momentum states. The energy gap in crystals is found to vary somewhat with direction of the wave vector because of this mixture of angular momentum states. However, the phrase ‘s-wave superconductivity’ is used to describe the state which belongs to the most isotropic representation. In a cubic crystal, the gap would have the same value along all  axes. Similarly, the phrase ‘p-wave superconductivity’ applies to crystals in which order parameters might have different signs in the  directions. The symmetry of the energy gap Δ(k) is said to be s-wave if it is isotropic, and p-wave if it has the symmetries of a p-wave orbital, and d-wave if it has the symmetries of a d-wave orbital. In 1987 SC was discovered in materials with planes of CuO2. The highest value of the transition temperature at ambient pressure is Tc = 133K in Hg–Ba–Ca–Cu–O. Although much is known about these materials, the basic mechanism causing SC at high T is still unexplained. Even the symmetry of the energy gap is still uncertain. Most experiments support the d-wave picture, but some still support the s-wave symmetry. The electron seemed to be paired in spin singlet states, with a very short coherence length.

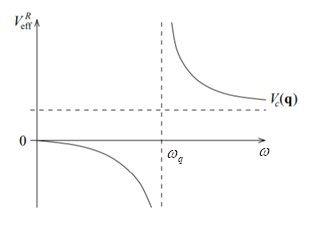
Note magnetic fields have an effect on SC’s. Large fields tend to make the electrons in the bound pair want to align with the field, and hence with each other. So as the field strength increases, there is a tendency for the bound pair to split, which would destroy the SC properties of the metal.

**BCS Model Interaction**

Let’s return to the effective e-e interaction from the normal metals folder. We found.



which looks like this, as a function of ω,



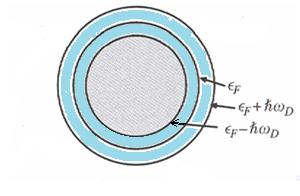
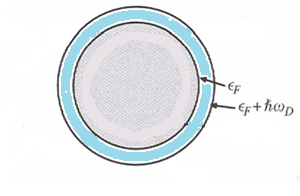
So one can derive all the superconductivity stuff using the full effective interaction, and this is *necessary* for electrons that are ‘strongly’ coupled to the phonons – as in the case of lead, at least. But for weak superconductors, one can get away with a simpler model. So q and ω represent, kind of, the momentum and energy *transfer* between two colliding electrons. So if our electrons are (k, εk) and (k´, εk´), then q = k´-k, and ω = εk´ -εk, or something like that. Anyway, if we’re mainly interested in the superconducting possibilities of our effective interaction, then we’ll confine our attention to the region where it becomes attractive, which is |ω| < ωq. And we’re most interested in the interaction between electrons on the Fermi surface, as these are the ones which determine most of the properties of the system. For interacting electrons on the Fermi surface, their momentum transfers can be large. For instance <q2> = <(k´­ - k)2> = <k´2> + <k>2 – 2<kk´> = <k´2> + <k>2 – 2<k><k´> = 2kF2, as we’d expect k and k´ to be uncorrelated. So <|q|> ~ (√2)kF. So average momentum transfer would be large, and so ωq ~ ωD, where ωD is the Debye frequency. So for electrons on the Fermi surface, we have: |εk´ - εk| < ωD. And if the electrons are around the Fermi surface, then this roughly tantamount to saying that for electron interacting, we need |εk| < ωD (before and after collision) So we might posit the following ‘effective’ Hamiltonian, insofar as we’re concerned about the superconducting properties of a metal,



where Vs(k,k´,k+q,k´-q) equals either of the following two options:



These two blue interpretations of the effective potential aren’t totally dissimilar. Only particles about the Fermi surface would have an appreciable probability of exchanging an amount of energy ωD, because only particles within that range would have enough phase space to scatter into to make the probability substantial. The two blue ranges are illustrated by picture below:

Finally, we’ll see that it is particles with opposite momenta and spin which tend to pair, at least for s-wave superconductors. Might argue that particles with equal and opposite momenta have strongest tendency to pair as their free states would be degenerate, and so more susceptible to a perturbation. Whatever. And so we may take any of these models and replace the interaction part with:



And so we have:



This is the model we’ll use most often. And in accord with our potential approximations above, we’d have Vs(k,k´) to be either of the two guys below:



Moral of the story seems to be that the BCS Hamiltonian is relatively ad hoc, and is just designed to get the qualitative physics right, and the different models will do that – just have to get the attractive interaction around the Fermi surface idea into the model.

**What I saw Mahan do, kind of**

I saw Mahan do something a little different though. In the effective interaction, he approximates ω with ξq = εq – μ, like we kind of do with first order approximations to self-energies. Then we have:



Since ωq < ωD << εF, this means that εq is restricted to the region around μ, i.e., around εF. So our q’s are too, at least if we want a negative interaction. If q is around the Fermi surface, then ωq ~ ωD, and so we could say,



And so we’d see that is negative for |ξq| < ωD, i.e., for εq within a range of ωD about μ = εF. We could simplify our potential further, using fact that ξq is close to zero, and therefore using the Thomas-Fermi approximation to the εRPA function,



And we could go even simpler:



(the θ function is like a Heaviside function and just returns 1 if the constraint is satisfied – otherwise 0) And accordingly we’d write H as:



Under this approximation, the position space potential would look like this:



Won’t bother to work this out per say. It will suffice to note that the potential will have a position space range of Δr ~ 1/Δq, where Δq is the range in momentum space (this follows from arguments identical to those made in many other contexts, like Heisenberg uncertainty principle, etc.). And so Δr ~ εF/kFωd. I think εF ~ 104kB and ωD ~ 102kB and so this would be like Δr ~ 100/kF. And 1/kF is around a lattice spacing [units…units]. So Δr is Δr ~ 100a. Although this range >> lattice spacing, that doesn’t stop anyone from making a delta function approximation to the potential [in other words setting the wavevector range of the interaction to ∞]. So sometimes we say:



Someone said this is essentially the model used by Cooper, and subsequently by Bardeen, Cooper, and Schrieffer in their theory of superconductivity. And then we could just write:



Moral of the story seems to be that the BCS Hamiltonian is relatively ad hoc, and is just designed to get the qualitative physics right, and the different models will do that – just have to get the attractive interaction around the Fermi surface idea into the model. Going back to position space (see QM/Many Identical Particles/2nd quantization position space file, this would read,



and if we wanted to simplify to an interaction between Cooper pairs, then this would go to:

