**Thermal Equilibrium Properties**

There is an easier way to get low T results. It’s called the Sommerfield expansion, and we don’t need to use fancy gamma function properties and such.

**Sommerfield Expansion (for low T)**

Let’s consider a general calculation of some quantities of interest of the following form.



where in second line we did some integration by parts. Since n´F(ε) is something like -δ(ε-μ) at low T, it should be accurate to expand H(ε) about μ. So,



Let’s fill in nF(ε),



We can extend the integral lower bound to -∞ w/o error, since while nF goes to 1 in that limit, its derivative goes to zero exponentially fast. So,



The integral we crossed out vanishes because ∂nF((ε+μ)/β)/∂ε is even (making integrand odd). Can see via:



and so, sans a constant which disappears upon differentiation, nF((x+μ)/β) is odd, which makes its derivative even. Or better, I suppose,



Now we have, quitting at 2nd order,



That integral is just a constant, which we could work out with contour integration, or just guess. It’s π or π2 something. Looks like it’s π2/6. So,



Of course we could keep going to higher orders in 1/β. I think this is called the Sommerfield expansion.

**Chemical Potential in low T limit**

Now we can use this to do a few calculations. Let’s get the chemical potential first. To do that, we use the formula for n = N/V.



And ρ includes spin degeneracy. So,



Now we have to solve for μ, preferably as a power series in (kBT)2. We could use reversion of series. So we’ll say μ = μ0 + (kBT)2μ1, plug this into the equation,



and then Taylor expand for small T, out to O(kBT)2,



and equate coefficients on both sides:



So we have:



It’ll be worthwhile to generalize this result beyond the case where ρ(ε) is the free particle density of states (or equivalent up to some effective mass). So starting over, leaving ρ(ε) general,



And ρ includes spin degeneracy. So,



Now we have to solve for μ, preferably as a power series in (kBT)2. We’ll do reversion of series again. So we’ll say μ = μ0 + (kBT)2μ1, plug this into the equation,



and then Taylor expand for small T, out to O(kBT)2,



and equate coefficients on both sides:



So we have:



If we fill in our free ρ(ε), this reproduces what we had earlier.

**Internal Energy, heat capacity, pressure in low T limit**

Now let’s look at the internal energy (density). We have:



Now,



So,



Now fill in our result for μ, ultimately keeping terms only to O(kBT)2,



Well,



So now we have:



So,



It follows that the heat capacity is:



Borrowing following formula from the Free Day electrons file,



we can say,



Can get the pressure,



So,

