**Ferromagnets**

Now going put the inhomogeneous MFT’s in a new light, as we did for the homogeneous ones, by using the HS identity to rigorously map the discrete partition function to a functional integral. All we have to do is use a more general Hubbard-Stratonovich identity. So,

**Hubbard-Stratonovich approach to MFT on Ising Model (long λ limit)**

Again consider our Hamiltonian in an inhomogeneous field:



Forming the partition function, over Sz eigenstates we have,



Here we’ve subsumed the β appearing in Z into the coupling constant, J, and h:



K is an N by N matrix whose elements give the interaction strength between the i, j elements. For instance, if we were dealing with a 1D line of spins, and z = 4, then an ‘isotropic’ K would look like,



**Writing as a functional integral**

Last time we pretty much assumed K was a constant, and that h was constant. But now we won’t; we’ll allow spatial fluctions, as we’d expect there to be. We’ll use the Hubbard-Stratonovich identity, which is just a certain discrete path integral (see Path Integrals file).



So then, we can write, ignoring the prefactor, which should make negligible contribution to the free energy in the large N limit,



Again, the spin sum can be done to obtain,



And now we have the functional integral representation.

**A check in the uniform field case**

Like before, we want to perform a saddle point approximation. First we simplify matters by falling back to a uniform field, and constant K matrix over z nearest neighbors. So we take the derivatives and find, kind of like before,



which is nothing but the previously determined mean field theory! So nothing is fundamentally different. And we see that the saddle point approximation always delivers to us the MFT.

**Going to continuum limit and keeping only long wavelength (small k) terms**

So we’ll go back to our action,



And we’re going to simplify it a bit. We want to investigate long wavelength correlations between the order parameter. Let’s write S as:



where the Rj are the positions of the lattice sites. Now let’s write it in terms of a discrete Fourier transform:



(the sum over k goes over just the BZ of the lattice) Furthermore, we’ll presume the interaction Kij to only exist between nearest neighbors (and be constant among them). We’ll also presume inversion symmetry so that ΣnnΔ**R** = 0. And finally we’ll presume the small **k** (long wavelength) approximation is good enough (close to the critical point it is good enough – all other terms can be shown, using RG techniques, to be irrelevant in that limit).



Let’s also just presume we’re in a cubic lattice, for simplicity. Then nearest neighbors are at:



So, presuming z = 6 nearest neighbors then,



Filling this in,



Note the sum over kis over **k** ∈ entire BZ = (2πnx/Lx, 2πny/Ly, 2πnz/Lz), nj is an integer running from -Lj/a to Lj/a. And *a* is the lattice spacing of our cubic lattice. And k is the magnitude of **k**. Let’s try to go back to position space now. Well I’ll do each part separately,



Now for the second part,



Now normally **k** in Fourier space ~ ∂/∂**r** in position space. So we expect to relate the **k**φ to something like ∂φ/∂**r**. In discrete position space, derivative translate to difference. So let’s just look at:



In the small *ax,y,z* = *a* << λ = 2π/k limit, we can say,



Now look at:



This means we can write the S1b term as:



Now for the other guy,



Now we’ll fill in K(k),



So we have, altogether,



And in continuum limit,



Could integrate by parts, and then we have, using n0 = 1/ad = N/V,



**Small φ expansion near the critical point**

Now let’s make an expansion near the critical point. Since φ ~ m, this means small φ, and we’ll also presume small h ~ j. Now let’s Taylor expand the ln(cosh()) function.



So we can say,



Terms neglected are irrelevant close to the critical point, as per RG analysis. Okay then altogether,



Now define, as we have before,



Also going to keep just first non-vanishing order terms as approach critical point. So 1-2Kz → -1. And we get:



**Mean Field/Saddle Point Solution**

At this point, we might attempt to evaluate the partition function by using the saddle point method. So we’d look for the φ(x) which minimizes the functional S[φ], and come up, formally, with some solution,



and then plug this back into Z, to say,



And then we could get the Free Energy,



where φ0(x) is implicitly a function of T and h. Another way to frame this is to say that we have a free energy *functional*,



where φ(x) is a parameter, kind of like the magnetization, as we saw in the previous files, that can take on non-equilibrium values. And the free energy *proper* (at least the Mean Field Free energy) is what we get when we minimize this.



Well let’s work this out a little. So we have to do:



where used some IBP. So this implicitly determines φ(x) in terms of T and h(x). And so

then we have:



such that:



What is the magnetization, or well, the magnetic moment per unit cell? This is,



But we know the thing in brackets is zero. So that leaves us with:



where we use fact that near critical point, βJz = Jz/kTc = 1 (there’s no point in keeping βJz as we are limited to the near Tc region, and I think if we hadn’t done the expansion near critical point, we would’ve found m(x) = φ0(x) since that’s what we got when we got in the Weiss file). So we can write the equilibrium free energy, if we want, as:



And we have the commensurate equation of state,



Might compare this to what we get when we examine the inhomogeneous Weiss Free Energy. The equation of state was:



In the limit of the critical point, where T → Tc and so βJz → 1, these match. I guess that’s all we’re looking for.

**Critical Point Properties**

Let’s look at the free energy close to the critical point. Specifically, we’ll look at the free energy per unit cell. For a homogeneous magnetization, in zero field, we’d have:



Gonna specialize to the zero field free energy. Then we can borrow our result for m near below the critical point (see work below) m = √(-r/u). Then we have:



The heat capacity is c = -T∂F2/∂T2. So we have:



which is what we got in the Weiss homogeneous file. For T > Tc we just get 0 (at least in the vicinity of Tc). And this corresponds to a critical exponent α = 0. Let’s look at the magnetization close to the critical point. I’ll presume a homogeneous field for now, and I’ll let h be non-zero. So then our equation would go to:



This is a cubic equation, whose solution is known. So we have from somewhere,



Therefore, supposing r to be negative,



Might observe this takes the form,



where,



in accordance with Widom’s scaling hypothesis (see Thermodynamics/Critical Exponents). We immediately see that,



… wasn’t so immediate. We’d choose the root that is aligned with j in the j→0 limit. There is a non-zero solution to this equation only when r < 0, and when it is, we have: m = √(-r/u). So we recover our critical temperature Tc = Jz, and critical exponent β = ½. We can also work out m(r = 0, j), to get δ.



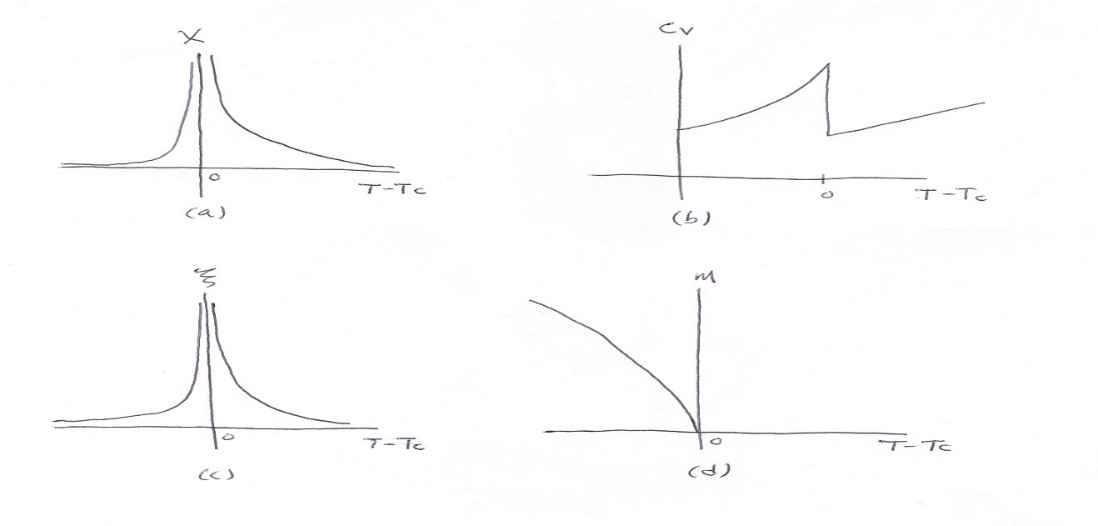
So clearly δ = 3. So we have,



But now that we’re allowing a spatially varying field, there are more correlation functions to be interested in (the top two guys below). But we’ll also observe that we can get the γ critical exponent too from these (see Thermodynamics folder/Phase Transitions file for derivation of last relationship)



Look I made a graph.



The Green’s function is the spatially inhomogeneous generalization of the susceptibility. The correlation length, which is inside the Green’s function, is basically measuring how long spins remain influenced by a local spike in the magnetic field. Often the correlation length diverges near the critical point, which is why it’s written the way it is. And ν is another relevant critical exponent then. We can get the equation for the susceptibility from the equation of state for instance,



Since m(x) appears in the equation, we cannot solve it alone. Clearly it’d have to supplemented by the equation for m(x) itself as well. But we can make a decent approximation, near the critical point, by approximating m(x) as its uniform mean field value m(x) ≈ m = (-r/u)θ(T<Tc). It seems that this is the ‘mean field’ way to handle m(x). And this approximation should be increasingly good, close to Tc. And do take note that this m(x) is the average magnetization itself. So it doesn’t make any sense to say <m(x)m(x)> here, as the m(x) here is not a random variable.



Can solve this equation by taking the FT of both sides,



which is,



Assuming r isn’t zero, we may take the inverse Fourier transform now,



where c = 2,1 depending on whether we’re below,above the transition. So we have:



So then we see that ν = ½ which is what we get for MFT, even though in reality it’s closer to 2/3 in most three-dimensional critical systems. And also see that η = 0, at least in 3D, as we are. And we also see that γ = 1 for us, in accordance with our previous calculations. So,



and we should get the same thing in all dimensions. Can compare these mean field exponents to the exact results,

Table

Description automatically generated

Now we’ll see how well they satisfy the critical exponent relations.

**Comparing to critical exponent scaling relations**

So we have found,



In the Thermodynamics file, we found/hypothesized the following critical exponent relations:



We can see that all of the equations are satisfied by our MF exponents. What about the hyperscaling relation?



This is satisfied in 4D alone, which is the upper critical dimension where MFT begins to work, exactly. MFT also works in higher dimensions, and gives the same critical exponents. But evidently the *hyperscaling* relation fails for d > 4. Nonetheless, it’s supposed to be valid for d < 4. So the fact that our MFT results fail to satisfy the hyperscaling relationship too means that our exponents cannot be correct in 3D (or 2D or 1D).

**Landau’s Phenomenological MFT of Phase Transition**

Going back to the free energy functional,



we’ll observe that is a more phenomenological way, explained by Landau, to obtain the same free energy functional. It’s used mainly for evaluating around the 2nd order critical point. It is based on a power series expansion of the free energy (functional) in terms of the order parameter (or order parameters) for the transition of interest. It assumes that the order parameter is ‘small’ so that only the lowest terms required by symmetry (and to keep the energy from diverging) are kept. The form of a Landau phenomenological free energy (functional) is determined entirely by the nature of the broken symmetry of the ordered phase, i.e., by combination of the order parameters that are left invariant under symmetry operations of the interaction Hamiltonian, and of course it is assumed that this would bear out from a more detailed analysis of the fully interacting Hamiltonian [this assumption can be justified somewhat by an RG analysis]. The undisputed usefulness of Landau theory rests in its simplicity – most of its predictions can be determined by solving simple algebraic equations. It is, of course most useful in the vicinity of second – order phase transitions, where the order parameter is guaranteed to be small. It, however, can be used with care to treat first order transitions, where there are discontinuous changes in order parameters, or to determine properties of ordered phases rather than of phase transitions themselves. In any event, we propose the phenomenological F:



where *m* is some parameter to make the units come out to energy, and T is temperature, h(x) is field, and φ(x) can be thought of as an internal parameter that characterizes possible non-equilibrium states. We can kind of think of it as the magnetization in this case. Kind of ignoring our work above, we’d would presumably get this F by calculating Z[T,h(x)], which would naturally be written as a sum over states Σstatese-βE(h(x),states), and the sum over states could be related to a sum over magnetization densities φ(x) ~ m(x), and so we’d get something like, Z = ∫D[φ(x)]e-βE(h(x),φ(x)) ~ ∫D[φ(x)]e-βF(T,h(x)|φ(x))] [there would probably be either constraints on φ, or multiplicities associated with a given state [φ(x)] that would have to be accounted for as well, which is what would change E → F] where F is an effective free energy functional.

Anyway, in our present case, the gradient term in our F allows for inhomogeneity and describes the tendency for neighboring spins to want to align. And the last term describes the coupling of the magnetization to the field. Now the interaction part of F, i.e. fL, must be invariant w/r to time-reversal. Since φ changes sign under time reversal, fL must be invariant under φ → -φ. Thus f must contain only even powers of the order parameter. Furthermore, close to the transition, where magnetization is small, at least on average, we would think we can make Taylor series expansion of fL and stop at low order.



In previous discrete theories we will have noticed that the r was typically something like r = a(T-Tc) and we’ll presume so here in order for things to match up to the homogeneous case. But in order for the partition function to be well defined, fL must be positive definite at large values of φ, since Z = e-F. Given that r can be negative sometimes, this implies we need to go out to fourth order and moreover that u must be positive definite. Integrating by parts on the ∇φ·∇φ guys, we can write this as:



We’ll find we can similarly reason out the free energy functional of a superconductor. That’s called the Ginsburg-Landau free energy.