**Weiss Inhomogeneous model**

Now we’re going to take an approach that will allow us to study the influence of non-uniform magnetic fields within our magnet. Let’s consider again consider our Hamiltonian in an inhomogeneous field:



J is an N by N matrix whose elements give the interaction strength between the i, j elements. For instance, if we were dealing with a 1D line of spins, and z = 4, then an ‘isotropic’ J would look like,



and hi can also vary with index/position.

**Mean field approximation**

So we can still make a mean field approximation. Split σ up into its average and its fluctuation about it. But this time, we’ll let the average, be position dependent, i.e., the local average.



where we’ve neglected fluctuation squared terms. Now we can do the sum over spins, kind of,



So then our Free energy would be:



Filling in heff again, dropping the 2 no one cares about (poor 2), then:



And remember that is position/index-dependent, so can’t do those sums yet. But now we can also get the equation for i. I guess it’s best if we think of Z as:



Then can see that we have N independent systems, which are yet, position-dependent. And so the average spin at a given spot, i, would be:



So we have:



which is the natural generalization of our previous position-independent result. Let’s put F in terms of the magnetization. So we’ll say,



With our equation of state, we can rewrite our F a little more nicely. Recall,



So,



So can say,



and hence,



hi is still there, implicitly, within the mi. And our equation of state is:



Note this magnetization is actually the (average) magnetic moment per site. So it’s *not* the magnetic moment per unit volume, i.e., the magnetization, as customarily defined.

**Going to continuum limit and keeping only long wavelength (small k) terms**

Now let’s go to the continuum limit. But we will have to restrict our consideration to variations which are small on the length scale of the lattice spacing between spins. And for the sake of simplicity, we’ll take our lattice to be cubic so that *a*a = *a*y = *a*z = *a*. Now let’s introduce the discrete Fourier transofrms of the spins on the lattice sites, as well as that of the interaction matrix J.



(the sum over k goes over just the BZ of the lattice) Furthermore, we’ll presume the interaction Jij to only exist between nearest neighbors (and be constant among them). We’ll also presume inversion symmetry so that ΣnnΔ**R** = 0. And finally we’ll presume the small **k** (long wavelength) approximation is good enough (close to the critical point it is good enough – all other terms can be shown, using RG techniques, to be irrelevant in that limit).



Let’s also just presume we’re in a cubic lattice, for simplicity. Then nearest neighbors are at:



So, presuming z = 6 nearest neighbors then,



Filling this in,



Note the sum over kis over **k** ∈ entire BZ = (2πnx/Lx, 2πny/Ly, 2πnz/Lz), nj is an integer running from -Lj/a to Lj/a. And *a* is the lattice spacing of our cubic lattice. And k is the magnitude of **k**. Let’s try to go back to position space now. Well I’ll do each part separately,



Now for the second part,



Now normally **k** in Fourier space ~ ∂/∂**r** in position space. So we expect to relate the **k**φ to something like ∂φ/∂**r**. In discrete position space, derivative translate to difference. So let’s just look at:



In the small *ax,y,z* = *a* << λ = 2π/k limit, we can say,



Now look at:



This means we can write the F1b term as:



Now for the other guy,



So that was easy. So our entire F is:



Now we can convert this to an integral, having especially in mind that we’re in the small *a* limit, so that we can approximate the deltas as differentials, and we’re dividing by lattice volume since each site should just count as 1.



Can say density n0 = 1/ad. Can integrate by parts in the second term.



Again the h(r) is implicit within the m(r). Let’s quickly, using our result above, translate the equation of state into the continuum,



And going to continuum limit,



**Small m expansion near the critical point**

So let’s specialize to close to the critical point, where σ will be small. Then using,



the free energy will go to:



So we have:



Doing the same to the equation of state, using tanh(u) = u – u3/3 + …



where we kept just the m3 term in the last cubic term. Keeping just these terms is commensurate with what we kept in the F expression. We’ll see this better later. So,

defining,



then we can write:



I’ll defer a critical point analysis till we get to the HS file.