**Liquid Helium Dynamics**

Want to consider the derivation of the sound wave modes in liquid helium. We can revisit our work in the Thermodynamics folder for this, specifically the Navier Stokes file, where we derived the non-equilibrium equations of motion of a fluid and worked out the wave equation apropos sound waves, and also the Plasma Dynamics file where we worked out the equations of motion of a two-component fluid (positive/negative charge). So we’ll split the fluid into the normal part (fluid n) and the condensate (fluid c). These are distinguishable and so the total entropy is just the sum of each’s entropy: s = sn(εn,**℘**n,nn,ψ) + sc(εc,**℘**c,nc,ψ). Note ℘ shows up in s here because we do not have impurities which would try to ‘relax’ the velocity distribution of any local piece of our fluid back to zero, and so it should constitute a real equilibrium thermodynamic variable. And I’ll let ψ be some constant external field, gravity I guess, which makes ψ = gy probably, and q = m. Also, unlike the plasma case in the Thermodynamics folder, I’m going to eliminate momentum and energy transfer between species, as well as heat transfer. So then I’ll say the balances for particle species n and c are, respectively,

 

(where **℘**i = mni**v**i and i = species n or c) And the entropy balance would look like this:



where s(**r**,t) is sn(**r**,t) + sc(**r**,t). We cross out the **v**csc term because the condensate does not carry entropy. By the way, ∇·(**℘v**) = ∇j·(℘ivj) (using Einstein summation notation) and ∇·**π** = ∇jπij = ∇jπji (‘cause π is a symmetric tensor), and ∇·(**π**·**v**) = ∇iπijvj = ∇jπijvi, again, ‘cause **π** is a symmetric tensor. There was some question in my mind, but proceeding analysis indicates we can write εn = un + (1/2)ρnvn2, and εc = uc + (1/2)ρcvc2, where un,c are the internal energies (densities), and the other guys the obvious kinetic energy densities. In fact, we will explicitly invoke this fact when we get to the discussion on the chemical potential, and it relates to the chemical potential when the fluid is at rest and in zero field.

**Skip this part basically – just working out conditions that make equations reversible**

Let’s see how the equations are modified presuming reversible dynamics. I’ll do this the way I’ve done it every time before, in the Thermodynamics file. So we’ll plug the balances into the entropy balance. For the normal part we’ll also use the usual equations: Tnsn = (εn + pn - **℘**n·**v**n – μnnn), and dsn = (1/Tn)dεn – (**v**n/Tn)·d**℘**n - (μn/Tn)dnn – (mn/Tn)dψ (where the moment mn = mnn = ρn is the mass density of the normal part). For the condensate part of the fluid we’ll have: 0 = (εc + pc - **℘**c·**v**c – μcnc), and 0 = (1/Tc)dεc – (**v**c/Tc)·d**℘**c - (μc/Tc)dnc – (mc/Tc)dψ, where we’ve eliminated entropy terms because it doesn’t carry entropy. From here out, we’ll presume the chemical potentials of the two species are identical, as should be since they are interspersed with each other. I guess I’ll have to split this up into LHS and RHS. So, LHS is given by….using Einstein summation notation by the way:



and the RHS,



Now solving for sint … and simplifying,



where each of the colored terms adds to zero. And we can simplify the π terms, and also going to change indices i→j and j→i in the first two of those ℘ terms (species n)…



Then can see thoes ℘ terms (blue) cancel with each other as well. This leaves us with (adding a δij to the species n ∇v term to make it match with the other one):



We can write the top line in vector language as:



Let’s write πcij = pcδij + (πcij-pcδij) to get:



with 0 = (εc + pc - **℘**c·**v**c – μnc), and 0 = (1/Tc)dεc – (**v**c/Tc)·d**℘**c - (μ/Tc)dnc – (mc/Tc)dψ,where mc = mnc, we can write this as:



So these all cancel out, and we have:



So clearly reversible dynamics simply requires **π**n,c = pn,c**1**. I guess that’s somewhat obvious in retrospect. There was probably a more succinct way to derive this too.

**Reversible Equations**

We’ll now specialize to the case of zero dissipation, i.e., reversible dynamics (so basically our liquid is always in thermal equilibrium of sorts) So our equations are now, setting **π**n,c = pn,c**1**:

 

And entropy balance is:



These, along with the definitions we used above:

Tnsn = (εn + pn - **℘**n·**v**n – μnnn)

dsn = (1/Tn)dεn – (**v**n/Tn)·d**℘**n - (μn/Tn)dnn – (mn/Tn)dψ

0 = (εc + pc - **℘**c·**v**c – μcnc)

0 = (1/Tc)dεc – (**v**c/Tc)·d**℘**c - (μc/Tc)dnc – (mc/Tc)dψ

are all we need to get the sound wave equations. But in practice, out of the last four, we will only need the Gibbs-Duhem equation, which is a consequence of these.



To proceed, the first step is to take the balance equations and add them together. If we combine the two continuity equations, and multiply by m, we have:



So,



Likewise we can combine the two momentum balance equations to get:



Guess I’ll leave it there:



And can combine the energy equations,



and so we have:



Then the entropy equation:



We can use just these four, along with the Gibbs-Duhem equation, to get where we’re going, but want to combine them to get a particularly useful equation relating **v**c and μ. Consider the steps we took towards getting the Euler equation, back in the Thermodynamics Folder/Navier-Stokes file, and apply them to the condensate. We used the continuity equation and momentum balance equations.



Now we’ll work out the momentum balance (using Einstein summation notation), and fill in the continuity equation at propitious moments:



and this is Euler’s equation:



Now just going to make a cosmetic change. Use the Gibbs-Duhem equation for the condensate (see Thermodynamics/More Potentials), which is easily derived from the equation for s: Ts = (ε + p - **℘**·**v** – μn), and its differential ds = (1/T)dε – (**v**/T)·d**℘** - (μ/T)dn – (m/T)dψ, where m = mn.



Now of course we have no entropy in the condensate, so this goes to:



and since this holds in any direction, we have:



and filling this into the Euler equation we get:



This isn’t the way this is usually written. Our chemical potential up above is that of the condensate, in the field, and moving with velocity vc. Usually we write this in terms of the chemical potential of the condensate by itself with no field and at rest. Can see Thermodynamics Folder/More Potentials for this, but the relationship between these two chemical potentials is:



So filling this in,



And so we have,



Now we want to use these *blue* equations, along with the Gibbs-Duhem equation, to derive the sound wave equations. We’re going to take a perturbative approach as usual. So we presume our thermodynamic quantities make only small deviations from their constant rest values. So velocity v = v­0 + v1 = 0 + v1­­; (mass) density ρ = ρ0 + ρ1; pressure p = p0 + p1; entropy s = s0 + s1; and temperature T = T0 + T1. And we will consider our equations only up to first order in smallness. Also, gravity should be a much smaller force than the pressure gradient, so we’ll ignore it. We’ll need to expand terms like **℘** to first order. This would be **℘** = ρ**v** =(ρ0 + ρ1)(**v**0 + **v**1) = ρ0**v**0 + (ρ0**v**1 + ρ1**v**0), which are the zeroth and first order terms. But note v0 = 0, so this is just **℘** = ρ0**v**1. Likewise, to first order, we can say: s**v** = s0**v**1. And also obverve that terms like **℘v** are necessarily second order at least. Another thing, quantities like ∂**℘**/∂t or ∇**℘** necessarily simplify to: ∂**℘**1/∂t or ∇**℘**1 because **℘**0 is constant. And same with all the other quantities. With that in mind, the first of our equations, linearized, is:



And also remember that ρ**v** = ρn**v**n + ρc**v**c. OK next equation,



As for the third equation, we have:



And finally our last equation, in place of the energy equation, is:



So altogether, our linearized equations are:



**The (First) Sound Wave Equation**

Now we’re finally going to use the peach equations, along with the Gibbs-Duhem guy, to get the (first) sound wave equation. First sound has to do with density/pressure oscillations (note these are canonically conjugate variables) and so we’re going to try to get an equation for how density perturbations (or pressure perturbations) propagate through the material. So take the time derivative of the first (linearized) equation, and the divergence of the second (linearized) equation



These two clearly imply,



Now we need to connect p and ρ. So p generally depends on ρ and T. Or another way to say it, ρ generally depends on p and T. But for a liquid, it is a good approximation that ρ only depends on p. And equivalently, that p only depends on density. And it depends on density through the bulk modulus, defined below:



And this is equivalent to:



So we can solve for p to first order,



and then fill this into our wave equation to get:



So we see the existence of density waves, i.e., sound waves, with the usual velocity v = √(B/ρ0).

**The (Second) Sound Wave Equation**

Now let’s go on to the second wave equation. The second one has to do with fluctuations in entropy, or temperature (note these are canonically conjugate variables), and how perturbations in these propagate through the fluid. More specifically, we’ll be dealing with entropy per unit mass, which I’ll denote with an underscore s = s/ρ. We’ll be using our peach linearized total balance equations, along with the Gibbs-Duhem equation equation. So let’s start with the latter. We have:



and drop the field again, and linearize the equation. This will give us:



(just fill in p = p­0 + p1, T = T0 + T1, etc., and we’ll see) Now since μ differs from μ(0) by just v2 terms, they are equivalent for us, to linear order. So can say,



Also keep in mind that μ is the same for both normal fluid and condensate and so even though we set s = 0 in the Gibbs-Duhem equation to get the m∂vc/∂t = -∇μ(0) equation, we aren’t restricted to s = 0). Okay now take these two of our linearized balance equations,



and plug them into the linearized Gibbs-Duhem equation. We get:



Now take divergence of both sides:



Now we want to put the RHS in terms of s. Now s generally depends on T and p. But for a liquid, it’s a good assumption to say that it only depends on T. Or equivalently, that T only depends on s. Well, actually, I think T would depend only on s (i.e., entropy per particle number - see Thermodynamics Folder/solid potentials). If so, then we can write (Einstein summation):



Now recall (derivatives taken at constant V or p, and N)



where C is the heat capacity. Not gonna make a distinction between Cv and Cp because there isn’t much of one for a liquid, and c is specific heat capacity. So we have the identity:



Now we need to convert the LHS to a double time derivative on s1. Hmmmm….so go back to two of our linearized balance equations, expanding out the latter in terms of normal and condensate components.



Since s1 = (s/ρ)1 = [(s0 + s1)/(ρ0 + ρ1)]1 = (s0/ρ0)[(1+s1/s0)(1-ρ1/ρ0)]1 = (s0/ρ0)(s1/s0 – ρ1/ρ0) = s0(s1/s0 – ρ1/ρ0), we have:



Note we can say s0n = s0 because of course the normal liquid contains all the entropy, and also, the volume we’re dividing by here in s0n = S0n/V and s0 = S0/V is a standard differential volume V = d3r, not the volume of the normal liquid itself, or the entire liquid itself. Another temporal derivative says:



And so the LHS of our prospective wave equation is now:



Putting the LHS and RHS together we have:



which is:



So these wave propagate with speed v = √((ρ0c/ρ0n)·(T0/c))·s0. The speed of (second) sound is apparently not constant with temperature. Shamelessly borrowing a guy’s picture:

Graphical user interface

Description automatically generated

I’m thinking the two speeds correspond to these two points on the excitation spectrum curve, which can be, at least for a small locus of points, be described by equation ε = pv. The first one, v = c, is the low T speed, and the second v = cmin is the higher T speed. The first one is clearly due to phonons, and the second to ‘rotons’.

Chart, line chart

Description automatically generated

Liquid Helium is a very good heat conductor. As there is no dissipation below the critical temperature, heat conducts via sound waves, specifically, second sound. So it is used to cool things like superconductors in the LHC for instance.