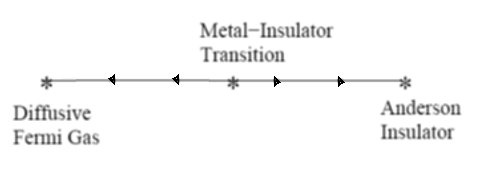
**Nonlinear Sigma Model of disorder**

The self-cnnsistent theory seems to be a sort of mean field theory, and so can access points around the phase transition, but not the transition itself. The single-parameter scaling theory seems to be true, but is inherently phenomenological, and would require the self-consistent theory to estimate the scaling function near the critical point. The NLsM is a field theory that we can use, like was done to analyze the FM phase transition. It can handle, indirectly, the fluctuations near the critical point. From it we can derive the β(g) scaling equations (in Q1D, 2+ε dimensions?) It seems to be mainly applicable in the diffusive regime (because apparently we expand in powers of small kurvatures),

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| --- |
| Are the higher k-terms relevant perturbations in 3D? I imagine they are and grow in importance, but can be considered small in 2+ε dimensions, i.e., proportional to ε? |

but one guy says it’s good up to ℓkF = 1. Also says one can derive probability distributions? Really how?



We can imagine an RG flow like the following, at least in 3D. If W > Wc then we flow towards insulator. Else, towards conductor (diffusive gas). W, = 0 is also a fixed point. Or we could say if g < gc then we flow towards insulator, and if g > gc, then towards conductor. In this sense g functions like temperature in a normal phase transition. σ itself would function as the order parameter since its zero in the insulating phase (thermodynamic limit), and finite in the ordered phase.

Earlier we had already written down an expression for the fermionic action. But disorder averaging is what complicates things here. And further, we need disorder averaged GF’s, not a disorder averaged F or whatever. So it’s this which complicates things. Anyway, we start with:



(not doing a thermal average so no μ) I’m going to presume spinless fermions, which ought to be OK, except for when it’s not. We can change variables to the temporal Fourier transform, and if we’re just interested in the GF’s in frequency space, we have:



Some quantities of interest, like the density of states, can be expressed in terms of a single particle GFR(1,1). Others, like the diffusion coefficient, conductivity, etc., can be expressed in terms of a product of two single particle GF’s like was done for the conductivity (recall how σ is of the form GFR(1,2)GFA(2,1)). The two-particle case can be obtained by functional differentiation of the following action (the extra term goes to zero because it involves average of a separate ψR, ψA)



where action is:



Now we need to consider how to take the disorder average of lnZ. To do this, first we use the following identity, called the ‘replica trick’.



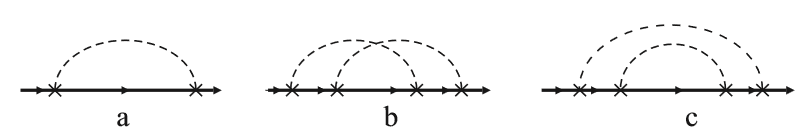
where

And then for the disorder average, we’ll presume a Gaussian weight factor: τ is the m.f.t, and ρ0 is the density of states. These factors are chosen to make the exponent unitless as required.



Note how the disorder averaging has introduced a sort of interaction potential among the fermions. Have to look more into this, but I’d presume this will now reproduce the GF rules, with the impurity circles and such, that we derived earlier --- somehow. Can also see that the interaction is a four-vertex term, which would only give us the following type diagrams,



which is what we found used the direct white-noise disorder averaging model when first discussing the disorder Hamiltonian.

Now we want to use the Hubbard-Stratonovich identity to separate out the interaction…so we introduce a square matrix Q with dimension equal to that of Ψ,



and use the following identity (at least you can see it’s true for Q = 1×1 matrix (and I’ve verified it’s true for 2×2)). Note Q is unitless.



Note we can neglect the prefactor because that concerns the matrix A, which constitutes the identity Tr(Q2) = QAQ, and A is independent of ψ. So this gives us:



Then, upon performing the ψ integral, we get:



Now the det is an important part which can’t be neglected. So to work out what it’s equal to, we can make the following manipulations…The determinant of an operator, O, is defined as the product of the eigenvalues λ of the operator in [ ].



Maybe if this doesn’t exist,, then determinant doesn’t either…Then we can make the following formal manipulations, and in the last step, we can use whichever basis we choose to evaluate the trace. Note that in our expression we Tr over the matrix elements and over position space.



And so we can write, formally at least:



Well, turns out we’ve neglected some symmetries in Q, which stem from symmetries in Ψ. If we had separated Q out into its true d.o.f., and carefully done the integration, we would’ve seen there is a ½ in front of the ln…and now I’ll evaluate this at ωR = E + ω/2, ωA = E – ω/2, which is what everyone does.



It seems that Q represents, in some sense, Ψ. Since we want an effective field theory for long wavelength behavior of Ψ, we will look for a spatially homogeneous (small k) saddle point solution to the functional, and expand the action about that point. Souer we look for the saddle point solution at ω = η = 0. Taking derivative w/r to Q, we get:



I get the following the result if I convert to momentum space, replace the measure with πρ0 /τ (because ddk is related to d.o.s., but its units are 1/Ld, and so yeah that makes it work out kind of), and do a sort of residue thing about the pole at E (presumably EF).

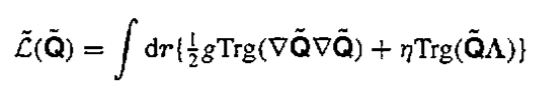


Maybe say integral can be approximated by value near Fermi surface, ε = E,



But should’ve gotten Q2 + 1 = 0 I think. Skipping to the end, we basically substitute Q = 1 + W, where W are fluctuations about that point, back into lnZ. Expanding the Tr(ln()) thing in small powers of W, which effectively means we’re looking at weak fluctuations of inhomogeneity (so weak disorder kind of I think?), we get this…Seen both forms:





And yeah we’ll recognize that prefactor on Q2 as g? Well can’t be exactly. It is a constant after all. With this action, we can reproduce the standard weak localization results, and can perform standard RG analysis to get β(g), and work out the scaling exponents consequently. But again, this seems to require that we have weak disorder W = 1/kFℓ << 1, due to the expansion in the Trln() term. And this consequently seems to restrict the validity of the RG analysis to d = 2 + ε dimensions. But he says that we can extend model to 1/kFℓ ~ 1 regime too….?

**C. Nayak NLsM notes**

We may make the observation that we’re dealing with a sort of critical phenomenon. 1 and 2 dimensional systems always scale towards a localized state. The 3 dimensional case seems to be marginal. Actually even the 2+ε dimensional case is marginal. Therefore 2D seems to be the lower critical dimension. Moreover, we seem to have a quasi-scaling variable, lng (or g depending on your preference). In 1D, 2D, g always scales to 0. In 3D we have the following ‘RG’ flow. Fixed points are a clean gas, diffusive Fermi gas, and Anderson insulator. If we start with any of the 3, when we scale the length, we’ll of course remain with what we started in. On the otherhand, if we have a little bit of disorder, then as we scale the length we’ll go towards the diffusive Fermi gas. At the critical disorder (the metal-insulator transition) we’ll remain fixed as we scale the length. And then as we add still more disorder, we’ll scale towards the Anderson insulator fixed point when we increase the length. So we will sort of think of the high n (impurity concentration) case as the disordered phase, and the low n case as the ordered phase. So metallic is ordered and localized is disordered – phase.



But what is the scaling parameter, analogous to temperature, which is ‘flowing’ towards the fixed points? We are hypothesizing that it is g. We might ask what our corresponding order parameter is and it would seem that it is something related to σ as it is 0 in the insulating phase and non-zero in the conducting (ordered) phase. Well, the order parameter is supposed to be finite (1) in the ordered phase and so we look to the formula for σ = e2Dρ(E), and postulate that it is ρ(E) which is our order parameter perhaps. This is somewhat born out by the NLσM of disorder, but the problem with this is that it turns out that it doesn’t change as disorder is increased past the critical point b/c ρ(E) is analytic across the critical point. So rather, it is perhaps the *local* density of states which is the order parameter. This makes sense as this would capture the the localization behavior of the electrons and also because it should be related to the # of open/closed channels we’ll discuss later. Thus, it would be related to γ12 which seems to function as the order parameter judging from the GDMPK results.

In any event, from Thoules argument, we should have that g is the control parameter, analogous to Temperature. This is given further weight, and cast into familiar language when we write an effective field theory for the slow modes of the disorder system. The basic observation that we wish to recognize is that it is only long wavelength excitations that survive the whole disorder averaging process are density-density, spin-spin type excitations. Particle-hole excitations are not long wavelengthed simply because they decay exponentially since the green’s function g(r,ω) ~ exp(-r/2ℓ). I suppose that density-density ones could be (though they must also decay exponentially no?) because localized electrons will act like a lattice and thus the primary type of excitation would be density waves? We would like to conjure an effective field theory describing the interaction between these modes. So we will write an action that describes the density-density excitations – I guess. So we have a typical action for single particle excitations as:



with the impurity potential inside. Then we want to disorder average over the possible V’s. A question would be, ‘how should we weight the different V’s?’. A Guassian normalization is probably the most convenient. Another question would be, disorder average what? Z, F, else, or are they all equivalent procedures. I suppose we must disorder average over F. In Kevin’s class, we disorder averaged H. And this would be equivalent to disorder averaging F. So we form,



Now a problem is that F = lnZ doesn’t have a functional integral representation, though of course Z does. But we get around this by using the identity,



And so our problem is relatively solved, since now to get lnZ, we can just calculate Zn/n. And Zn will just be, as with any exponential, the multiplication of n copies of our Z together. So we’ll perform this procedure, assuming a whole number n, and then take the limit n → 0 and hope for the best. So then we obtain,



The V can be integrated out, which then couples the ψ’s together to give us,



where,



Observe how the disorder averaging produces a sort of interaction between the fermions.

And now we’d be in position to apply the RG stuff we examined before.

Now let’s write an action-sort of-for the density density correlation function. To wit, consider just the density:



We have to disorder average each GF separately, I think. So we’ll have a different effective action for each disorder averaged GF. I don’t think we can make a generic disorder averaged S from which ‘derivatives’ will give us the disorder averaged G. So we have in mind to calculate the j-j correlation function. This is basically a product of two particle-hole GF’s.

Note the equality holds b/c the latter < > is the complex conjugate of the former <>. Such quantities, as you might recall in Kevin’s notes, were encountered when we were trying to get the j-j correlation function. So in the functional integration formalism, this is just the product of the retarded action integral, with the advanced action integral. But we can combine them into one integral of course. And we get,



where,



Note the way way we handle the poles in the action, SE,E´ gives us the retarded/advanced GF. But now he takes the



term out of the expression. I think we’re just focusing on the action, trying to solve it roughly speaking, and then we’ll straightaway have an expression for the GF’s w/o having to resort to a perturbative analysis. And now we look to disorder average by forming F ≈ Zn, and then disorder averaging. So gluing together the n copies we get,



where,



And then disorder averaging we have,



with,



Note how the disorder averaging has coupled the advanced and retarded parts together. Now streamline the notation a bit, defining,



Then we can write S as,



where,

At this point we could apply perturbation theory to the perturbation term – the 4 coupling term at the end. But it is a non-marginal perturbation, and so we cannot expect sensible results term by term. We would have to sum an infinite # of diagrams. So to better deal with this term in the functional integral format, like we did with spin interactions, we’ll use a Hubbard Stratonovich transformation to decouple the interaction term. So to put it in the proper form before we do so, we write the last term as,



Then we introduce the HS field, decoupling the interaction, as,





Now we can integrate out the electrons, and come to,





(note that the 1/τ term wasn’t there before but was introduced via the HS decoupling)

The trace is over the 4 matrix elements of Q. Now we want to evaluate the integral via saddle point, we obtain the saddle point equation,



Assuming the solution is translationally invarariant, we get,



And now if we recall in the spin case, how we were able to relate the saddle point t with the m. So here too we can, and this implies an expression for the GF’s to be:



And next we want to look at fluctuations around the saddle point. So expanding S about Q = Λ, and defining, GRR = GR(E +ω/2), GAA = GA(E – ω/2), and GRA = GAR = 0, we have,



And then expanding the first ln to first order, remember that the terms linear δQ are 0, we get,



where we’re using the matrix notation,



and similarly for the Q’s. Now writing out the G’s and performing the integrals we get,



so note how the QRA, and QAR fields are non-massive, while the QRR and QAA fields do have ‘mass’. The RR and AA terms are the ‘longitudinal modes’, while the AR, and RA terms are the ‘transverse’ ones which conform to Goldstones theorem. If we continue to expand about the saddle point, we’ll acquire extra terms, which due to symmetry, must altogether conform to the following form.



And this action is almost identical to the NLσM in FM phase transitions.



In the former case, Q serves as the order parameter, and in the latter, it is **n** which does. Q is related to the density of states since,



And the imaginary part of GR is ~ spectral function ~ density of states per **k**, ρ(E).



But as is indicated above, the critical exponent, β = 0, since the density of states doesn’t actually vanish across the MI phase transition. It is always finite. On the other hand, n is only finite in the ordered phase – the spontaneously broken symmetry phase. So one can say that in the MI transition, the symmetry is always broken. The lack of analogy between the MI case and the FM case has to do with taking the n (replica index) → 0 limit to complete the Replica trick. There are correspondances though.

1. In the ordered (metallic) phase there are massive longitudinal modes QRR, QAA, and **n**H where there is a mass gap between excitations states. The former is the diffusive propagator G ~ 1/(iω – Dk2). The latter is spin flips in the direction of **H** with GF ~ 1/(H + q2)
2. There is a critical point at d = 2 + ε. Before which there is no phase transition to the ordered phase (metallic in the former case, ferromagnetic in the latter).

Note that the renormalization group equation is for the coupling to the coefficient of the (∇Q)2 term which is DNF which is ~ σ by the Einstein relation which is ~ g in 2 + ε dimensions, which we seem to be in.