**Hubbard-Stratonovich MFT of Ising Model**

Heretofore, we have performed a MF analysis of our model, but have not generated it in a way in which is perspicuous enough to allow systematic corrections, or see under what conditions it is accurate enough. So we shall remedy that now. So let’s go back to (changed notation σ → S):



As you increase z, MFT becomes more exact since the sum of a large number of random spins becomes increasingly well described by their average. However, this assumes that the spins *are* random, and that they aren’t correlated over the volume of the system, say. This is why a MFT approach doesn’t work so well in describing the critial region – because spins/whatever do become significantly correlated over long length scales near the transition. On the other hand, even still, it does work well even at the critical point when d > 4 – when statistical averaging dominates the fluctuations. In any event, to also illustrate how MFT is exact in a certain long range limit, let’s also extend the range of the model, without making it blow up. So we will let the z → ∞. But we’ll have to make J → J/N, say, in this limit, to keep things physical (this like how when you model a solid as a N connected springs, when you take N → ∞, you have to take the spring stiffness to ∞ as you shrink the length of the spring to 0 so that the overall stiffness of the spring remains constant). So we’ll analyze:



(apparently, to get the limit to match with our previous theories, we’ll say J → 2J/N rather – this is why we have no ½ in front) Now let’s construct the partition function. Note each spin operator Si is really just Szi, and so we can take the trace of this Hamiltonian to evaluate the partition function, by inserting a complete set of many-body SzT eigenstates on either side. And this will turn it into,



where each of the Si’s can be . Now since,



we can replace the product spin operator with ST­. Doing this we get,



Calling ST to be S, and ignoring the constant that doesn’t matter, we have:



This is more concise, but still not easy to evaluate. This is where the Hubbard Stratonovich identity comes into play.

**Converting Z to functional integral**

Now we’ll convert to a functional integral. Recall identity,



Setting a = N/βJ and b = S, we have the ‘Hubbard-Stratonovich’ Identity:



And so we can write,



In the first case the many body states are coupled, and in the second they’re just independent spins. So its intereseting how one can ‘change’ the Hamiltonian, when purpose is to find partition function. So even if Hamiltonian isn’t solvable, it may be possible to still find the partition function. And once the partition function is determined, it may actually be possible to go back and find the exact many body energy levels. In any event, the sum can now be evaluated! and we get,



**Saddle point approximation**

Next we want to evaluate the integral via saddle point. And as we see, this should be exact in the large N limit. Now at some t0 (possibly multiple t0’s), this exponent will have a minimum/s, determined by:



And for T < Tc, there will be, as we saw earlier, multiple roots. The one with the smallest value of g will be the true thermodynamically stable root. The others would represent metastable states. Then we can write, expanding g about that point(s),



And so our free energy would be:



Let’s presume one root’s contribution predominates, as would be the case typically. Then we can say,



We can neglect the ln term, at least in the thermodynamic (large N) limit. Then we have:



So what is t0 and what is its physical significance? Taking the derivative and setting to 0 we get,



So this determines t0 in terms of T and h, and we’re technically done therefore. We have f.



We can compare to our MFT from before by putting this in terms of m though. So to make the association of t0 with m we use the equation of state:



Then we can get ∂t0/∂h implicitly…



and proceeding,



So there. And then our free energy becomes,



And our equation of state translates to:



which was our previous MFT equation of state! And of course everything, including m, is implicitly in terms of the canonical variables T, h. We can invert the equation of state, using,



to write,



which we’ll recognize as the same as our first (Weiss) mean field free energy, well except for the absence of z.

**Ergodicity Breaking in light of model**

Going back to a couple lines up, so if there are two minimums, how does the system choose which one to be in? With the field on, one minimum will be larger than the other, and so it will choose that one (eventually, though if carefully prepared, it could remain in that metastable state for a while). But with the field off, they’ll be the same, and so how will it choose?? Physically, this happens because of ergodicity breaking. The system will be thermodynamically exploring all of its available states. At the time of the transition, it will be at one end of the spectrum, and so will get caught at one or the other mimimum. On the other hand, we could say that h is never really zero, it will point in one way or another, even if infinitesimally, and that is the direction in which the magnetization will coalesce. Presumably these phase transitions occur in real life only for a large sample because of the necessary large time for the system to cycle between states in order for the system to be stuck in one ‘end’ of the spectrum. Mathematically, to make the transition occur, we must take the TD limit assuming some infinitesimal h. Then we have,



So this gives us,



Now we take the the large N limit and we see that (assuming positive h), the second term vanishes. So we can go back and simply continue on with our positive root. Doing so we have,



So observe that the transition only occurs mathematically in the thermodynamic limit, at some infinitesimal field. Also we see that large coordination number and large dimension are similar – we’ll have many interacting spins, letting the law of averages kick in. And so we see of one two ways in which MFT is exact.