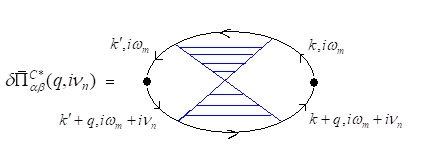
**Weak Localization (Quantum)**

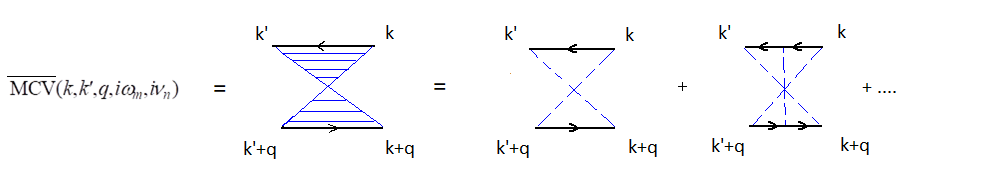
Now we take into account weak correlations, which would become more important as the impurity concentration is increased. So we decreasing ℓ now. Note that when ℓ becomes ~ the electron wavelength λF, then we cannot picture the electrons as diffusing through the sample, since the spatial extent of the e- ~ λF would be ~ the scattering length. Rather we would have to take into account the interference of the electron wavefunction with itself. The subleading contribution in 1/kF­ℓ - the so-called maximally crossed diagram –takes into account these interference effects. Apparently all crossed diagrams are of the same order in 1/kFℓ.

**Maximally crossed contribution to conductivity**

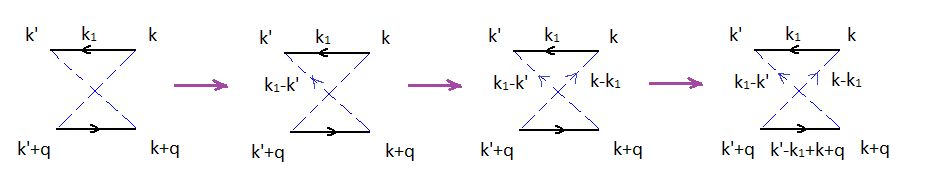
The general diagram is:



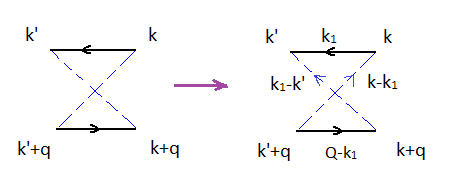
Energy momentum is flowing against the GF arrows, i.e., clockwise. And the GF’s themselves are resummed within the self-energy approximation, like before. Note the vertices are fully dressed with the diffuson (ladder vertex). The maximally crossed (MC) vertex is shown below. Note that the impurity lines are distinct. We only have one impurity per line. Leaving off the frequencies, this is what we’d have for the vertex.



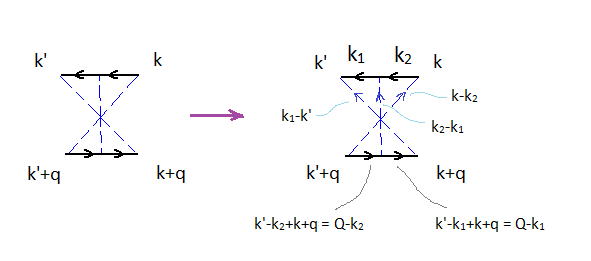
To work out the first diagram, this is what we could do. We label the top GF with k1.



Then that means momentum k1 – k´must be flowing up along the impurity line. Then since momentum k must exit the diagram at the top, a momentum k – k1 must flow up the second impurity line. Then the last emendation to the diagram follows. We will call Q = k + k´ + q. So then we can write the diagram as:



and likewise, the second diagram would be:



So we might notice then that MCV depends on k, k´, Q, and iωm, iνn. Anyway, so the current-current correlation function would then be:



and then the conductivity correction would be:



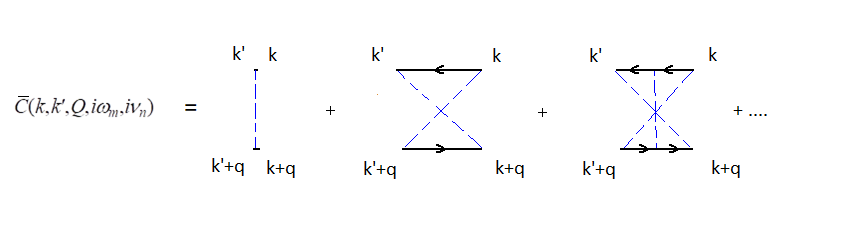
So let’s see how we can get MCV…

**The Cooperon**

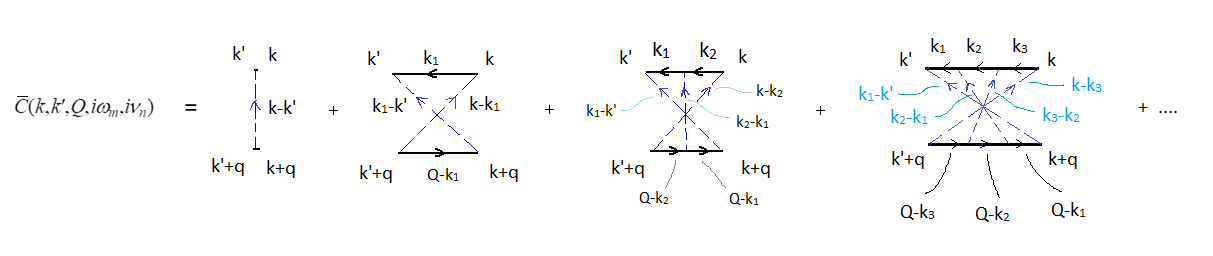
To get the maximally crossed vertex, it’s convenient to define it in terms of the Cooperon, which is equal to the maximally crossed vertex, with the addition of a single impurity line,



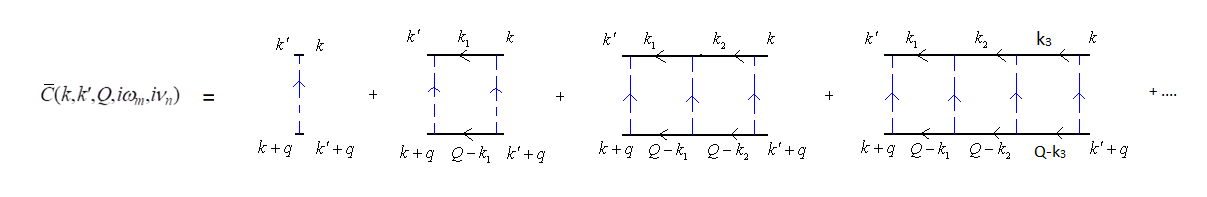
where, diagrammatically,



which is, writing in all the energy (well, leaving out energy) /momenta (and remember the energy/momentum lines are flowing in opposite direction to the charge arrows on the GF lines).



To facilitate writing a recursive equation for C, we’ll now ‘untwist’ the diagrams, and write them as so:



If we write out a couple terms, leaving integrals and frequencies implicit,



Then we can see that the following recursion relation holds … having trouble seeing it strictly from the diagram though….



**Formal solution for Cooperon with δ(x) potential**

Let’s evaluate the cooperon for a delta function potential.



and define



Then



We can see that the RHS is independent of k, the first argument of C, for this potential. *Thus C is independent of its 1st argument*. So now we can pull C out of the integrand and write



Now we can see that C is independent also of its 2nd argument, kʹ, as well – so C only depends on the total momentum, or, refering to the diagrams themselves, Q can be interpreted as a sort of momentum transfer. So finally we write:



where



**Calculating Cooperon in DC and low T limit**

Since C\* depends only on the magnitude of its first argument, and anticipating that we will be primarily interested in the small |Q| (he says that the largest contribution to the integral occurs when Q is small – though should note that none of the calculations we would need to do with large Q seem too difficult), small |νn| (small |νn|implies low temperature – recall that to obtain the DC conductivity we’ll perform iνn → ω + i0+ → ω → 0) limit, we approximate



where  is the angle between and Q. Now we’ll split our integral into an angular part and radial part. We’ll perform the radial integration first.



where



where

To simplify the contour integration, we'll approximate ρ(ω) by ρ(EF). Then if ωm and ωm + νn have the same sign, we can close the contour so that it encloses no poles → . If ωm < 0 and ωm + νn > 0 then we have (all these h’s should be replaced with 2π since I’m replacing h-bar with 1).



and,



While for ωm > 0 and ωm + νn < 0 we have



In either case we have,



There are other ways we could write up the sgn function of course. For |νnτ| << 1 and QνFτ = Qℓ << 1 we can summarize these results as



We could’ve done the integration exactly in all cases without the expansion, but in any case, integrating over and keeping only the first nonzero order terms in the expansion, we have…



where



is the d-dimensional diffusion coefficient. Recalling that for the zero range potential



We find that for |νn|τ << 1, ad DQ2τ << 1, we have:



and



So finally, sort of:



**Plugging Cooperon into the conductivity correlation function**

Putting this all back into our current-current correlation function, we have:



Having in mind the DC conductivity, we’ll be taking the q → 0 limit. And so:



where



(note Q = k + k´ + q = k + k´ cause we take q → 0 limit) We can drop the ni|Vi(k-kʹ)|2 term because this gives precisely the first rung of the non-crossing vertex correction that we showed vanish for the zero range potential (recall how τtr = τkF for the delta function potential – which implies that all rung additions, which gives the difference between the two scattering rates vanish). This is also why we have treated the dressed vertex as a bare vertex – for the delta function potential, they are the same because all vertex corrections vanish.



Now looking at the range of m in the summation for a given n, we have,





So we can write this as (note that n can’t be equal to zero – this would set the theta function equal to zero),



If |νn|τ << 1 this also means that |ωm| << 1/2τ, so we can set ωm = ηsgn(ωm) in each of the Green’s functions, which implies (these approximations ought to become exact when we take the limit iνn → ω + iη → iη to evaluate the DC conductivity, no?)



Since the summand is independent of m,



and therefore,



Now we will analytically continue via iνn → ω +iη.



And finally since we’ll want to take the limit that ω → 0 we’ll just keep the first term (since when we take the limit, it must be done from above or below 0, which will mean that one of the terms will always be zero). Or we could just say we’re considering positive frequencies only.



Now going back to:



we’re ready to calculate



We have:



Now owing to the diagonality of the conductivity tensor we’ll add up the diagonal components and divide by the dimension (leaving the ω limit implicit for the time being).



Let us now transform to a sort of ‘center of mass’ coordinate system: from k, k´ to

 → 

You can verify that the Jacobian of the transformation is 1. (I wonder if these would have any relation to the Mandelstam variables in QFT) Then we can write:



which we can write as



And I is defined as



Now we want to look at the small Q region, which we anticipated to contain the majority of the correction. So recalling that



we consider (note that to get the DC conductivity we want to set ω = 0,



Putting this back in the expression for the conductivity correction



Noting that



and



We get



For d ≥ 2, the integral diverges at large |Q|. However, we assumed in the evaluation of the cooperon that |Q|ℓ << 1. Therefore we can cut off the integral at |Q| ~ 1/ℓ.

**Dimension d = 3**

For d = 3, we can safely take ω → 0 to get



So



Since the Drude conductivity can be written (setting τtr = τ):



we see that the maximally crossed diagrams lead to a negligible correction



If we had included the small Q cutoff that we do below (though unnecessary here), we’d find that there is a correction inversely proportional to the min of the phase breaking length and system length. A final consideration: the mean free path ℓ is given by ℓ = vFτ, where



And so ℓ ∝ ni-1 and we see that as ni increases the the weak localization correction to the conductivity will increase. Though to completely shut it off the impurity density would have to be on the order of the lattice constant.

**Dimension d = 2**

For d ≤ 2, the Q integral diverges at small |Q| for ω → 0 (and large Q when d = 2). Physically, though, there is a small |Q| cutoff as well - the greater of

* The inverse system size L-1. The smallest non-zero value of that the momenta can take on in a periodic system is certainly 2π/L. But can’t it take on the value of zero as well? In a clamped system this would collapse the wavefunction to zero. So maybe not. And in such a system, there are no negative values of the momentum (such quantum numbers don’t give new wavefunctions). So in this case the momentum Q = k + kʹ would be constrained by this value. But I’m not sure this would hold for just a general periodic system.
* The inverse dephasing (or Thouless)Lφ-1. Lφ ~ T-P/2 is the distance over which the particles’ phase becomes randomized by inelastic collisions. The value of P is believed to vary between 3/2 and 3 depending on the dominant inelastic scattering mechanism (e.g. electron-electron, or electron-phonon processes)

So for d=2



(while he gets the result below – keeping ω finite I’m not sure how, or why)



Since



So we have that



Although the conductivity correction is of order 1/kFℓ compared to the Drude conductivity, it has a logarithmic factor that formally diverges for L → ∞. This is the phenomenon of weak localization.

**Dimension d = 1**

For d=1,



I’ve neglected the 1/ℓ upper limit since this integral converges in 1D for large Q, and in the spirit of things, I should therefore neglect it. In any event the lower bound gives the largest contribution by far. The result he gets below (again keeping ω finite) is:



Since



We have that

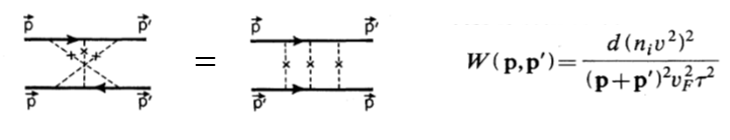


which is not a weak effect. This is a signature of Anderson localization in 1D (and 2D - see recent experiments): disordered metals become insulators as T → 0 (in which case Lφ → ∞) making L the minimum and causing the fractional correction going to 1 as L ≥ ℓ). Which would of course happen for almost any size sample. Physically, weak localization is result of coherent backscattering caused by constructive interference between pairs of particles traversing time reversed trajectories that return to the same point. In total, our exact results are as follows:

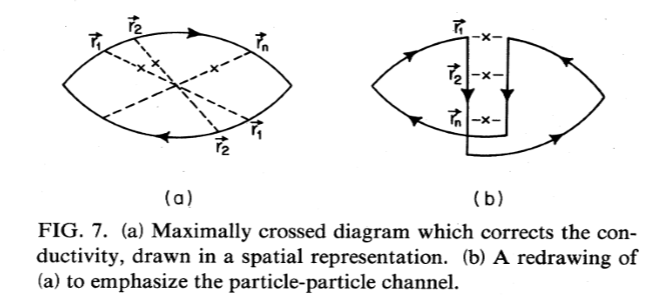


**Alternate Formalism**

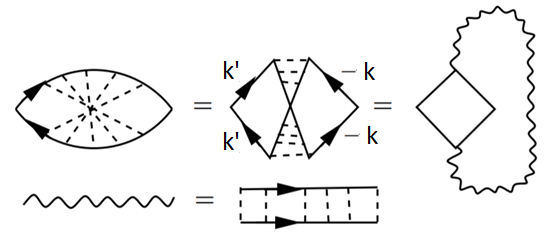
There is another way to write the WL correction diagrams, at least in the q = 0 limit? So we emphasize the Cooperon is related to the Diffuson, just ‘untwisted’. And this is called a particle-hole propagator? Particle-particle channel?



And we can write the maximally crossed diagrams like this:



Anyway, we can also represent them this way, with a so-called Hikami box.



With reference to the top diagram, we can clearly see the singularity where p = -pʹ. And P. Lee notes that the singularity is uncurable in d <= 2 I guess. And this describes 180⁰ backscattering, which is, interestingly, a 1D process (note this is also analogous to how TR paths make a singular contribution to the single particle GF – non disorder averaged). So it seems to suggest that localization is a fundamentally 1D process? But of course this is just the WL correction so who knows.

**Conductance Corrections**

We can solve for the conductance correction too. I’ll write it in a special form, that will be relevant later. Note I’ve used the approximate formula for σ0 = e2kFd-1ℓ, and so the formula here should be taken with salt grains…but of course we’ll find this to lay down with the predictions of the single parameter scaling theory. And it’s nice to see from the 3D formula that we need a minimum disorder for the localization length to make sense.



but when done correctly, we actually get:



**Putting in B field?**

So we already said that we could heuristically account for B’s by changing L to LB. I think this is only qualitatively correct. I think I find from P. Lee that diagrammatic calculations done with B fields produce the following results. Note it is somewhat ambiguous whether we’re talking about σxx or 1/ρxx – if you look at the Hall setup, we’ll note these to be different things.



where, ψ is the digamma function, and LB = √c/eB, which we heuristically encountered before. We’ll note that the 2D correction does bear some resemblence to the heuristic result suggested above, and semi-classically (i.e., the ln term). P. Lee also gives a Q1D result,



But this seems at odds with the fact that the conductance correction in is zero, from the DMPK equation? So I’m misunderstanding something here.

**Adding SO interaction**

In the presence of SO scattering and TRS (which can happen if B is absent because both S and L change sign under TRS, which would cancel out the net change), P. Lee says that for lengths larger than both ℓ and LSO = √DτSO, the conductivity has a ~ ln(L) increase in value – this is related to the change in sign of the associated scaling function β(g). Here’s the correction in Q1D to the conductance:



If both SO interaction and B fields are present, then not sure, but P. Lee says this has been worked out…

**Higher Order Corrections?**

P. Lee says that we could go to order 1/(kFℓ)2, but that these diagrams have no singular backscattering contributions (so no large L divergences); therefore our estimates of the localization lengths are essentially correct.