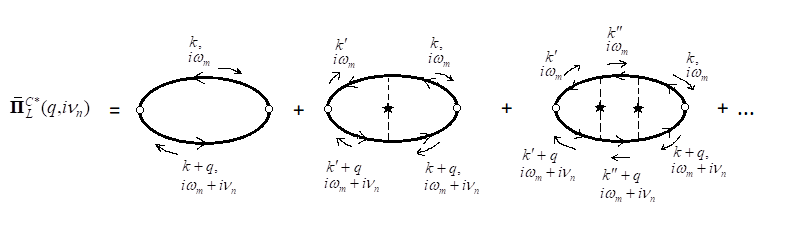
**Quantum Model**

It is impractical to sum all diagrams entering . The best that we can do is sum an infinite subset that we believe make the dominant contribution. Note that we’ll use our results for the actual Green’s function in our equation above.

**Ladder approximation to σ**

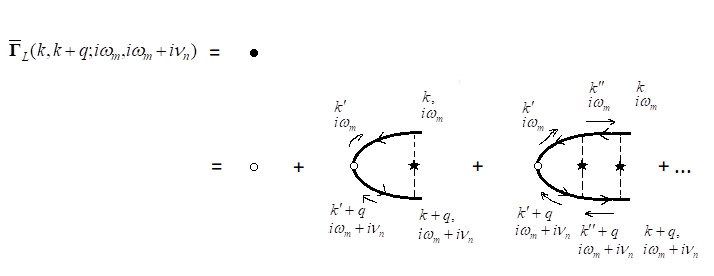
For instance, let us consider the Born limit in which only double scattering from each impurity is significant. Let us also neglect all diagrams with intersecting impurity lines, since they are suppressed by a factor of 1/kFℓ (but in some dimensions, do end up proportional to *sample* length, as it turns out). I imagine the reason why they’re smaller is the same as for why the crossed diagrams in the disorder averaged self-energy diagrams were small. Because the restriction that GF0(k) have k be near kF in order to be appreciable results in extra restrictions when the lines are crossed. So this would be the ‘weak’ scattering limit – called the ladder approximation.



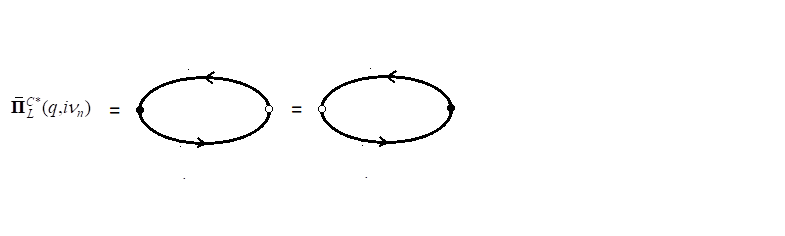
Note the momentum up through the last impurity line would be k-k´, and through the one behind that (if present) would be k´-k´´, etc. By the way, the GF’s in this approximation are ‘fully dressed’, as in they are already summed within the self-energy approximation in the disorder potential. For instance, something like this:



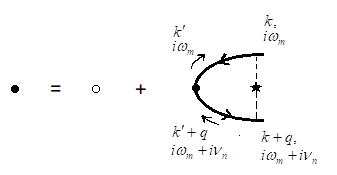
Let’s introduce the (ladder approximation to the) irreducible vertex (note it’s a vector) Γ. Note sometimes we’ll also refer to the vertex as γ, where Γ = eγ/m. γ is useful because it relates to the self-energy via various ‘Ward’ identities:



(the end energy-momenta coming out past the last impurity line is not included as a GF) In terms of it, we can write the correlation function in the ladder approximation:



And we can also write (diagrammatically) a recursive relation for the irreducible vertex,



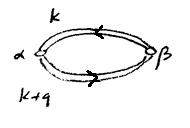
Note that the energy-momentum coming out of the filled in circle (irreducible current vertex) in the first diagram above is not the energy-momentum going out of the bare current vertex. There are a lot of ‘rungs’ between that momentum and the bare current vertex disguised inside the irreducible one. This self-consistent equation for the irreducible vertex can be written down as



But this is not easily solvable - you can’t Fourier transform this for instance because it isn’t a convolution. So we’ll just resort to using the first two terms in the expansion of the vertex. Then we’ll take (q,ω) → 0 as required, to get the DC conductivity. So first we’ll consider which is represented diagrammatically below.

**Zeroth order diagram:**

We’ll start off with the first order diagram. Note we’re going to we’re using the fully dressed GF here….so it’s not exactly ‘first’ order in the perturbation:





(remember 2 out front is due to sum over spins) where



Before we calculate all of this, let’s be aware of what we ultimately need for the DC conductivity. We will want to set iνn → ω + iη in order to analytically continue to the retarded Green’s function. Then we’ll want to let ω, q → 0 and focus on the imaginary part of  to calculate the DC conductivity.



So in these diagrams we’ll always encounter this Matsubara sum of the Green’s functions over the frequency ωm. We can evaluate this sum using the general method presented in the Stat Mech Math Appendix, and then we can perform the integral over the momentum. So a general Matsubara sum is (note that all frequency sums will have to be performed this way):



and so recalling,



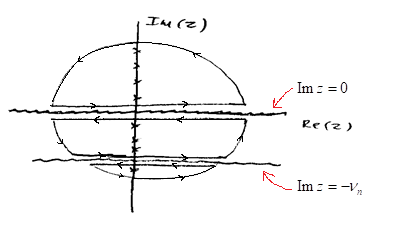
we have, preliminarily:



Now when we analytically continue the GC\*’s by doing iωm → ω, should note that the GF’s will have no poles in the complex plane, since when ω is in the u.h.p., our pole would be in the l.h.p., and vice versa. But these functions *are* non-analytic along the Imω = 0, and Im(ω+iνn) lines. And so we would have:



Now we can see that in the first GF we have a branch cut along the line running through Imz = 0 (i.e., the real axis). And in the second, a branch cut along the line running through Imz = -νn. From a more general perspective, there are branch cuts at Im(z) = 0 and Im(z) = -νn, because the general analytically continued GF, (k,ω), is discontinuous across the line Im(ω)=0 as we may recall from (k,ω+iη) – (k,ω-iη) = R(k,ω) – A(k,ω) = -2πi(k,ω). And the contour, with the appropriate branch cuts is shown below.



Luckily for us the branch cuts don’t fall on any fermionic Matsubara frequencies, only bosonic ones, so we're OK. Therefore, continuing with the solution…utilizing fact that circular part of contour goes to zero…



We shifted variables in each term to make the Green’s function along each branch cut be a function of (k,x) only. Note by the way that  so this simplifies to



At this point we’ll just focus on the part of the expression that gives us the DC conductivity. So let us evaluate



We have used here that A is real, and that



So now we have that (note that no approximations have been made yet in the evaluation of the diagram)…



Presuming we have an isotropic medium, we have:



So the first integral over k contains the vertex contributions. The n(x) comes from the Matsubara sums, and the two (k,x)’s come from the two Green’s functions. We’ll want to remember this form because we’ll be able to put the next order correction in a form proportional to this one. Anyway, now we can use the general form for



where



However for simplicity we’ll just take the limit A(k,ω) = δ(ω-ξk). Noting also that:



(obviously ‘cause it’s a step function at T = 0) we can calculate (in the T=0 limit) the lowest order contribution to the conductivity, but we’ll have to contend with the fact that A(k,ω)2 would seem to go to δ(ω-ξk)2 so we’ll have to be more careful about this limit. So, let’s calculate



using some Calculus of residues…So this implies that



So after taking all those limits, we have:



Now recall,



So,



and so (not sure what’s up with extra factor of π…gotta track that down sometime)



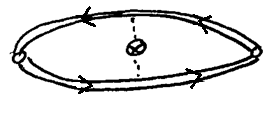
Some things to note:

Even though when expanded in terms of bare Green’s functions, consists of a sum of positive powers of ni, the conductivity σ(0) is proportional to τkF and hence to 1/ni. This shows the power of using the self energy to sum an infinite number of diagrams.

This equation has precisely the form of the Drude formula for the dc conductivity, but it has the wrong τ: it should not be the quasi-particle scattering lifetime, but the transport lifetime, τtr Since it matters what the k-state scatters into. If it scatters into a state nearly parallel to the original, then this would barely affect the conductivity, but if it scatters into a k-state antiparallel to the original k-state, then this would maximally degrade the conductivity. But our current formula takes no account of this difference. We will now see that the difference between τkF and τtr is accounted for by the vertex corrections

**First order diagram:**

Proceeding,



And this is:



where



The imaginary part operation on the product of Green’s functions comes from integrating around the branch cuts in the Green’s functions. Basically you have the Green’s functions minus their complex conjugates – hence the Im part. Once again we set iν­n → ω+iη, let (ω,q) → 0, and focus on the imaginary part of Π to calculate the DC conductivity.



Thus in our expression we’ll want to evaluate:



Now we’ll pass the Im operator through the integral,



And now since , we have



Then we set in the Green’s functions, and form the derivative in the Fermi function.



Now



Bearing in mind the extreme singularity of [A(k,x)]2 for τk-1 → 0 the first two terms dominate (unless |ξk – ξkʹ| < τk-1). This is because A(k,x), A(kʹ,x) will be centered about k, kʹ respectively and unless these two points are close together, the A’s will roughly cancel each other out since these Lorentzians will be peaked at different places. Recall that τk-1 is the width of A(k,x) and so our condition requires that the peak of A(kʹ,x) be roughly within the width of A(k,x), thus making them nearly on top of each other and therefore comparable to the first two terms which each is a Lorentzian on top of itself. And so our expression is modified to…



And now we’ll return to our expression for the conductivity.



We can swap k and k´ in the first of these terms because σ(1)σβ = δαβσ(1) and k, kʹ enter the σ(1) integral in symmetric fashion which implies:



Then presuming homogeneity, we have,



and then,



where,



Λ is the contribution of the rung in the diagram. So now σ(1) has the same form as σ(0), except for the addition of the factor Λ(k,x), which represents the vertex correction from the diagram. When we approximate the spectral and distribution functions as the appropriate delta functions, this will restrict our vertex correction to the Fermi surface – a constant independent of k and x, and thus our first order correction to the conductivity will be proportional (by the vertex correction) to the zeroth order contribution. Continuing on the vertex correction…



We wish to find out what the limit of this function is as the inverse lifetime goes to 0.



Thus



Note that 1/τkF ∝ ni so ni/τkF is constant in the limit ni → 0. Thus the vertex correction σ(1) is of the same order as σ(0). Approximating



we can use the multiplication property of delta functions again to write as



And the A(k,x) delta function will restrict k in Λ(k,0) to the Fermi surface. So finally we’ll have



where **k**F is a vector on the Fermi surface. Thus we’ll have,



And therefore

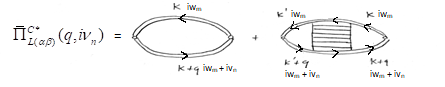


So to order 1, the conductivity is

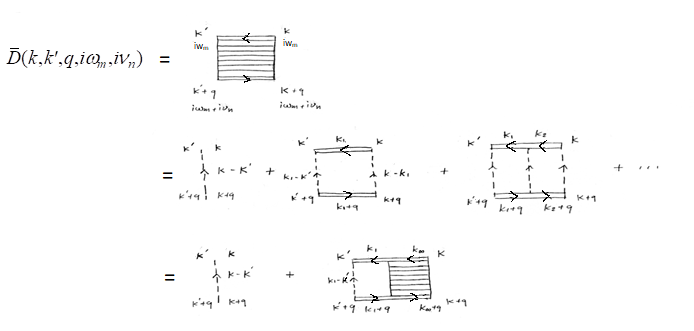


**Full ladder approximation**

Now move on to the full ladder diagram approximation, which includes all 'rungs' on the ladder, not just 1. Note that the impurity ‘star’ is suppressed on the impurity lines, and that each impurity line really consists of two lines, because of the 'star’ in the middle.



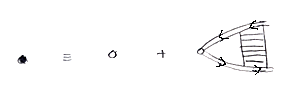
(momenta going against the GF arrows) where the set of all ladders is called the Diffuson: D(k,k´,q;iωm, iνn) (again momenta going against the GF arrows (but with arrows on impurity lines)



where I kind of left off the frequencies in some of the diagrams for convenience. The diffuson, D(k,k′, q, iωm, iνn) satisfies the recursion equation.



We can think of the sequential arrangement of impurity lines as sequential collisions with impurities – hence the descriptor: diffuson. In contrast, when we look at weak localization, the impurity lines will be crossed, suggesting correlations between collisions at different times. We can write the relationship between the diffuson and the (ladder approximation to the) irreducible vertex as, diagrammatically:



which is to say:



Anyway, the effect of summing over all diagrams - including all the ‘rungs’ in the ladder is simply to modify our formula via the change (which you well might guess)



We can simply multiply the Λ(k,ω)’s because they all have the same arguments. Remember we have taken q → 0. Therefore the ladder approximation to the conductivity is



And this will be equal to



where we recall,



and **k**F is some vector on the Fermi surface. This combines with the τkF to give us:



and if make, in the first term, k an element of Fermi surface at T = 0, then ξk = 0. So then we have:



and,



Note the difference between  and  is the weighting factor



where is the angle between the incoming and outgoing electrons. The weighting factor reflects the fact that back scattering degrades the current || to much more than forward scattering does. It is interesting to note that for a delta function potential Λ(k,ω) = 0, and thus there is no difference between the two: τkF-1 = τtr-1. Note that these calculations have been done at T = 0. If we wanted to go to finite T, we’d have to do a Sommerfield expansion of the distribution function.