**Excitations**

**GS of BCS 2-body Hamiltonian**

The model we’ll use is:



So the first thing to do is to investigate whether the effective interaction is capable of doing anything qualitatively different to the particles. So we’ll investigate the possibility of ‘bound’ states between electrons above the Fermi surface. By ‘bound’ state, we mean a state of the system whereby two electrons with energy ~ εF can pair to form a state with energy E < 2εF. We’re not looking for ‘real’ bound states whereby E = KEcm + Erel and Erel < 0 (and commensurately, the relative wavefunction is localized). Rather in our problem, as we’ll see, we’ll be working with two electrons at the Fermi surface with opposite momenta (and spins) and will have therefore E = KEcm + Erel = 0 + Erel. And we’ll simply find Erel to be < 2εF. Basically, we’re just asking whether the interaction potential will lower the energy spectrum of the particles *at all*. That this is possible is not foregone; consider the case of a single 3D particle in an attractive spherical well potential (see QM folder). Recall that when 3D interactions are weak there are no bound eigenstates and an attractive interaction won’t even affect the range of possible particle energies (meaning it would be the same range as we’d have for completely free particles), though it will affect the wavefunction, by scrunching it up within the well and creating a phase shift. But the energy isn’t affected because, roughly, even though the potential might tend to lower the wavefunctions energy inside the well, it will also scrunch the wavefunction up inside the well, increasing its kinetic energy. And so (curiously) the two effects will cancel, unless V is strong *enough*. And we want to see if things are different here. So we want to get the ground state of the two-particle Schrodinger equation, and consider its energy.



and near T = 0, we can just say μ = EF. As was done in the multiple distinct particles file, we can go to center of mass coordinates:



where M = m1 + m2 = 2m, and μ is of course the reduced mass. So we see that the Schrodinger equation splits into two independent parts – one giving the energy of the relative motion about the center of mass, and another giving the energy of the center of mass (which is purely kinetic). We can solve this equation using separation of variables. Clearly the solution is of the form,



It’s more convenient to use as the basis of the two-spin space the eigenfunctions of the total spin operator S2 = (S1 + S2)2 which are χs,m\_s = χ0,0 and χ 1,-1, χ1,0, χ1,1, shown below:



So we have:



Plugging this into our equation, we’d get:



The eigenfunctions of the relative equation we’ll call ψn(r), and the energies εn (n would stand for both radial and orbital degrees of freedom). Then we’d look for the ground state configuration of this pair of particles. We’d set K = 0 therefore. And this would suggest the two electrons have more or less opposite momenta, though this cannot be asserted unequivocally because the momentum operators do not commute with the Hamiltonian. Since they’re fermions, the ground state wavefunction must be anti-symmetric. So either the spatial or spin part must be anti-symmetric and the other part must be symmetric. From our analysis of particles in double wells [see QM folder], for example, we know that a symmetric spatial wavefunction would tend to have lower energy because it has lower curvature and would also tend to hover about r = 0 where the interaction potential would be strongest. So we’d postulate the spatial part to be so, and spin part to be anti-symmetric [another reason this choice is prejudiced is because we ‘know’ that these S-wave superconductors have no magnetic ground state properties, and so we want total spin quantum number to be 0, not 1]. Thus the spins would be likewise be anti-aligned, roughly. So we’d have:



So what is ψ0(r)? Need to solve:



The question is then, is there a solution to this equation with ε0 < 2εF? Since it is the Fourier transform of V that we know, makes sense to write ψ0(r) in terms of its Fourier transform,



And now take Fourier transform of both sides:



Now we’ll fill in our model of the potential, to get:



We can solve for ε0, thanks to the separability of the potential – and maybe this is one reason why people approximate it that way. Of course we presumed Vs(k,k´) was a function only of k-k´ in the preceding analysis, whereas we are now presuming it’s not, with our explicit model. But that’s what is done. So….



where we can multiply both sides by that θ outside the integral since its just 1, or its zero and doesn’t matter. Now integrate both sides w/r to d3q, and note the cancelation because the integration variables are dummy variables.



Performing the integral, taking advantage of fact that ωD << εF, we have:



Solving for ε0,



And this is approximately, taking λ to be small:



So we do get a ‘binding’ energy of sorts from the attraction. Might observe that we can’t ever get a ‘real’ bound state ε0 < 0. But the sort of binding that we *do* have is non-trivial. For instance, when we analyzed the 3D attractive spherical well potential states in the QM folder, we found that unless the potential strength were sufficiently strong, the potential would have no effect on the energy spectrum. Rather the states would all be free, and all have the characteristic positive energy spectrum k2/2m. The only effect of the potential would be shifting the phase of the free-particle eigenfunctions. So the fact that our potential here *does* affect energy spectrum (and does so for arbitrarily weak strength), even though the particles do remain fundamentally ‘free’, is significant. I suppose one may imagine that we get this ‘binding’ because the electrons have a reduced phase space to scatter into, thanks to the occupied Fermi sea, and so the interaction is able to ‘trap’ them. Can visualize the electron pair as orbiting each other (so at all times one has momentum k and the other -k).

**Note to Self**

I’ve seen other people do things a little differently, probably better in retrospect. They do like we did for the Kondo Effect GS wavefunction. They consider a GS Fermi wavefuction, up to some kF, and then they add a Cooper pair to it, and show that the energy is lowered, by the amount calculated above. This demonstrates that the free Fermion GS is unstable to the addition of Cooper Pairs, and so an entirely new GS must be constructed/hypothesized – probably one consisting of these Cooper pairs.