**Excitations**

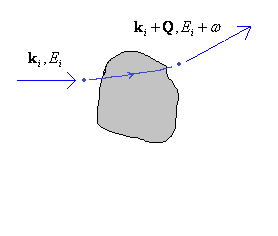
In the condensate current estimates are that at T = 1K each atom possesses KE ~ 15kB, and

PE ~ -22kB. From this we see that the average binding energy is ~ -7kB. These are the properties of the entire fluid, at T = 1, but at T = 1, the superfluid predominates. And so we can effectively take these to be the properties of the superluid.

This contrasts quite a bit from the estimate obtained from assuming no interactions, that KE ~ (3/2)kBT (3kB/2 at T = 1). We will be interested in studying the low lying excitations of 4He, to get a better understanding of its liquid properties (as it exists as a liquid, a superfluid at these low temperatures).

**Neutron Scattering: Working out energy spectrum experimentally**

One of the main techniques used to study such a liquid is neutron scattering experiments For instance, we consider a highly energetic neutrons (greater KE than the average KE of the 4He atoms) sent into the substance. It goes in with momentum **k**i, and energy Ei = ki2/2m and emerges with momentum **k**f = **k**i + **Q** and energy, Ef = kf2/2m = (ki + Q)2/2m = ki2/2m + ki·Q/m + Q2/2m = Ei + ω. This is shown pictorally below:



As discussed in the inelastic scattering file (QM folder), the energy-dependent scattering cross section would be:



where m is the mass of the neutron, Ξ the strength of the neutron/crystal (or superfluid) nucleus interaction delta function potential, N the number of such nuclei in the crystal (superfluid), and S(Q,ω) the Fourier transform of the time-dependent structure factor,

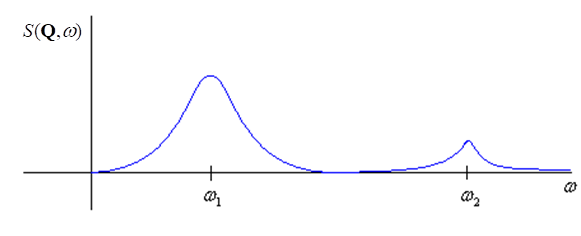


where < > is a thermal average, and H is the Hamiltonian for the quantum fluid itself. I guess we can write this as:



**Energy Spectrum**

To determine the energy spectrum experimentally, we would aim our detector in the direction of **Q**. And we’ll get a whole bunch of ω values. We can plot the probability distribution of ω’s. Presumably, the spectrum will look like a resonance(s),



Just like with scattering previously discussed in the QM folder, a resonance would be associated with a bound, or metastable state – something approximating an eigenstate. Therefore we’d infer that there is an eigenstate of the system with total momentum Q and energy ω1, and also another one with energy ω2. And of course there might be more resonances. Two such familiar examples would be where we can have particle excitations and plasma oscillations at the same momentum (frequency) but different energies. We would continue to do this for all of our Q’s, and develop a curve ω1(**q**), and ω2(**q**), etc. These curves would be the excitation spectra. The spectra for He4 is given below:



The dashed line is what we’d expect of single particle excitations. The upper three solid lines are, I’m assuming, acoustic wave excitations. They are all longitudinal. They are supposed to start out with velocity υs = 240m/s or so (so that ωk = υsk). The dips in the acoustic spectra are apparently from interactions with roton bound states. The first dip is a single – roton bound state. The second dip is a two roton bound state, etc.? At higher k, he says that will all join the single particle excitation curve (dashed line). So low k excitations seem to be density waves, and high k excitations particle-like.

**Momentum Distribution function and Condensate fraction**

We can also use the scattering cross section to determine the momentum distribution function. And we’d like to determine the # of particles in the condensate. So we just have to figure out how to relate S(**Q**,ω) to these things. To start, we go back to the definition of S and expand the R’s about t = 0. The argument is that since neutron scattering happens quickly we only need the short time response of the R’s. Using the Heisenberg eqn. of motion for the operators, we can solve for **R**(t). We’ll recall from QM this is simply,



where H is the Hamiltonian for the fluid. We need only go out to second order, say, for reasons to be elaborated. Presuming an H like H = ΣiPi2/2m + Σi<jΦ(ri-rj), we know that this will just come out to (since operators follow classical equations of motion):



where **F**i = ∇iΦ is a function of all the particle positions. Now we plug this into S,



Now Mahan says that we’ll be presuming Q and ω very large. If so then the latter assumption means we can neglect higher order terms in t, as those terms will wildly oscillate and wash out to zero. So we’ll neglect the force term. Another reason for this is that we should expect that a thermodynamic average of the force is roughly zero since the particles aren’t making any net displacements. Nonetheless, we are keeping information about the force – because the force information is embedded inside the < > which is a thermal average over the exact states. So then we can say,



Then using the Feynman theorem,



we can write,



where Q is magnitude of **Q**. Since r’s commute, we could combine the two r terms into the same exponent. But then large Q assumption allows us to presume only i = j terms will for the analogous reason that otherwise i ≠ j terms will wildly oscillate with Q and not make any meaningfull net contribution. So,



In terms of the wavefunctions, the expectation is:



But we’ll recall (from many-body correlations file in QM), modified for the case of absent spin, that:



So we have:



oh and the sum Σi just gives us N. The time integral then gives us:



where n0 = N/V. Now Mahan says that S(Q,ω), where ω is the energy transfer Ef – Ei, is an odd function in ω? And so makes the modification…



Whatever. So we have,



So we see that we can plug in a theoretical momentum distribution and see how well it predicts the scattering cross section. Or, we can try to deduce nk from the scattering cross section. And we will perhaps also want to determine the fraction of particles in the condensate. To do this, we split the momentum distribution into condensate/regular parts.



Plugging this expression into S we have the two terms as,



Then we want to perform the k-integral so we split it into an energy-angular integral. We may presume nk depends on k only through the energy. So align our k coordinate system with Q. And then we can say,



The two angular integrals will give us one, as long as:



Call the former lower bound k-(Q,ω), the latter k+(Q,ω). So we can write



Differentiating w/r to ω, we get (the 1/(Q/m) comes from chain rule) (and ignoring the δ part?):



and so we can kind of determine nk from experiment. A typical nk plot is shown below.



Neutron scattering experiments therefore seem suggest the simple form for n(k) as a Gaussian, peaked macroscopically at k = 0. And experimental results show that the Gaussian does fit the empirical data well.



where N0 is just the number of particles in the k = 0 state. Note that N0 will be comprised of both superfluid and normal fluid particles. And also, not all superfluid particles will be in the k = 0 state; thanks to the residual kinetic energy of the superfluid state, the superfluid particles will be spread across a range of k-states. Same too of course, with the normal fluid. Still, as T → 0, more particles will ‘drop’ into the k = 0 state, and I think particles in the condensate are especially prone to do so. I think current estimates (Mahan) are that N0/N → about 10% as T → 0. Unlike classical ideal gasses, where *w* = kT (*w* is the parameter in the exponential above), we find from fitting to neutron scattering experiments that *w* ~ 10K for low T (it must be around kT for high T). The kinetic energy per particle can be calculated from this form easily.



And thus <KE> ≈ (3/2)*w* ≈ 15K (near T ~ 0, 1, when N0/N ~ 0.10) as discussed above.