**Conduction**

I’m just going to look at the conductivity through the lens of the Boltzman equation. But we could use diagrammatic techniques too. Might refer to the electron-phonon folder for a Cliff’s Notes version of this option.

**Conductivity via Boltzman Equation**

We’re going to carry out the calculation, presuming an acoustic spectrum of phonons, presuming, i.e., that the electrons themselves screen the ions’ vibrations, and make the e-ion interaction short ranged. So we’ll be using effective e-phonon interaction (see Interaction file),



In the TF approximation,  reduced to:



as we saw in the Interaction file. This is equivalent to using the unrenormalized interaction vertex, g, with a δ function potential and the renormalized eigenfrequencies ωq in place of the unrenormalized ones: Ωq. Absorbing the 1/√V into the vertex, and doing the sum over λ, we can write:



Note that Ξ has units of energy (in faux Natural + Gaussian units). And we will approximate the ωq using the Debye model, which presumes a linear spectrum ωq = cq, out a maximum ‘Debye’ value of ωD = kDc (has to be a maximum, recall, so that we have only 3N acoustic modes):



For comparison, the Fermi momentum and energy are, in a 3D sample,:



So kD ~ kF. But c ~ 100m/s, and vF ~ 100km/s, so ω­D << EF. Note c is just the speed of the acoustic waves, certainly not that of light. Again we look to see how the E field will affect the distribution of electrons in the sample. In an external electric field exerting force F = eE, the distribution function of the electrons, subject to scattering by the e-ph interaction, will follow the discrete version of the Boltzman equation (see Stat Mech/Classical NESM Boltzman 1PI,2PI) for the occupation number nkσ(r). To first order in scattering this will be:



Note that the customary simplification of the collision integral for interactions which have time-reversal symmetry doesn’t apply here, since the interaction isn’t TRS. And of course we’re going to look for the steady-state non-equilibrium solution to this equation. But we need to update it a bit to take account of phonon processes. Looks like the proper way to do this is as follows. We always start in a state with some set {Nq} phonons, and a single electron. And we look for scattering into our electron state|{Nq}>|p´> → |{N´q}>|p>, and then also scattering out of our electron state|{Nq}>|p> → |{N´q}>|p´>. Then we do a thermal average over {Nq}. This average will be T-dependent of course. This is a little different than I would’ve thought. I would’ve thought we’d do |{N´q}>|p´> → |{Nq}>|p>, and then also scattering |{Nq}>|p> → |{N´q}>|p´>. But nope. Few more points. I’m going to elide the spin variables, and the λ = 1, 2 phonon polarizations, as the spin variable and transverse phonon polarizations are just spectator variables. So we’ll have something like, for the RHS [if we went to higher order of Veph in W, then multiple phonon scattering processes would be allowed, but as it is, we can see that Veph to the first power will only allow single phonon creation/destruction]:



The sums are over all p´ values, and additionally all occupation numbers N´q for those all possible q values. Also note E{Nq} = ΣqωqNq. Now we’ll note the following matrix element:



And the modulus is,



The cross-terms are zero because we cannot simultaneously satisfy both sets of delta functions. So plugging these into our collision integral,



Now there is a sum over Nq´ = 0, 1, 2, 3, …, ∞, for every q. The delta functions will collapse these sums. And we’ll have:



We can associate each of these four terms with a Feynman diagram,

Diagram, schematic

Description automatically generated

The top row describes scattering into state |p> via destruction/creation of a photon. And the bottom row describes scattering into state |p´> via the same (if go back to beginning, can see first terms in top/bottom row correspond to *a*, and second terms in these rows correspond to *a*†). I guess I’ll group them by delta function before moving on (taking advantage of fact Mp = M-p):



**Now let’s examine the high T regime (T >> ωD)**

So starting with:



Assuming that T >> ω­D (where ωD is the large q asymptotic energy of the phonon dispersion curve, you’ll recall), we can greatly simplify the collision integral. In this case, the majority of phonons will have momenta around qD ~ kF­, and energy ωD << T (it will still be most probable to have lower momenta, but the expectation of the momenta will be ~ qD I think, since the bose distribution will be so broad). Consequently electron momentum relaxation will be fast, since the electrons will have momenta around kF, but their energy relaxation will be slow, because their energy will be around T >> ωD. Moreover, two typical electrons will have an energy difference >> ωD. Can estimate this classically. At high T, electrons become classical and follow a Maxwell distribution p(E) ~ ρ(E)e-E/kT = √Ee-E/kT. And normalized p(E) ~ √Ee-E/kT/(kT)3/2. Then <E> ~ ∫dE p(E)·E ~ kT, and <E2> ~ ∫dE p(E)·E2 ~ (kT)2. So then, accounting for difference in the exact values of these integrals, var(E) ~ (kT)2 → std(E) ~ kT. So we may presume that energy differences between two given electrons are around kT and this is much greater than ωD as aforementioned. So we may neglect ωD in comparison with εp – εp´. This makes the electron scattering roughly elastic, and simplifies our integral to:



Now the phonon occupation numbers can be simplified by neglecting 1 in favor of N in the N+1 terms. And we expect Np = N-p. So this simplifies things to:



Now we’ll take the thermodynamic average, as we mentioned before. The rough average phonon occupation number is:



So now we may say,



This form is exactly the same as the impurity scattering case, where,



with:



And the solution to the Boltzman equation in the impurity case was, in the low Temperature limit:



and so the result follows in our case here that:



we can do a little work on the inverse lifetime.



where we note that since εk = εq, |k| = |q| and so that simplifies the denominator of cos(**k**,**q**) a little bit. And also used fact that **k**·**k**´ is odd in k´ which makes its contribution to integral vanish. So we have:



And now for σ, it likewise follows from the impurity calculation that:



where c is the speed of the acoustic waves. Proceeding,



Now nF(ε) = 1/(eβ(ε-μ)+1) of course. For high T, ωD << T << EF (we keep the restriction T << EF ‘cause otherwise solid would probably melt) we can presume nF to be θ(E-EF), and so we’ll get:



And so ρ ~ T for high/roomish temperatures. This makes sense ‘cause higher T should be equivalent to higher ‘disorder’. What if we did go higher, presuming it didn’t melt? We need to know what μ does for high T. Going back to the statistical mechanics file, and from the work on the Fermi gas, we have:



So for large T we have:



Going back,



Now -μβ → -lnβ → ∞ in the small β limit. In that case we can safely approximate 1/(eε + 1) →

e-ε. So,



So this would go as 1/T3/2, but it’s probably not even a solid anymore.

**Now let’s examine the low T regime (T << ωD)**

Going back to:



Now we cannot neglect the ωq’s in the delta functions, and we have fully inelastic scattering going on. Some rough estimates are, assuming an acoustic spectrum again, the majority of phonons will have energies ωq ~ T, and momenta therefore around q = ωq/c = T/c, FWIW. And the electron energies, will as before, differ by around a thermal energy, εp – εpʹ ~ T. Therefore the energy transfers will be comparable and we cannot neglect ωq in favor of εp – ε’p. Also certainly cannot say that Nq + 1 ≈ Nq. Apparently it is too difficult to work out σ, so we’ll satisfy ourselves with getting the relaxation time. If we analyze the work above, we’ll see that this will at least give us the proper temperature dependence. We can get the relaxation time by comparing the RHS Boltzman equation with the RHS relaxation time approximation equation:



So we can see that -δRHS/δn|n = n\_leq, would, for instance, give us 1/τ. So let’s do that. Going back to the collision integral (wrote it a little differently – but still equivalent since Mp = M-p, and expect Nq = N-q):



and taking derivative w/r to nk, and implicitly evaluating n’s at equilibrium, we have:



Then simplifying a little more,



I guess we can ignore the δ(0) terms. Maybe since M(0) = 0 anyway….? Proceeding,



Maybe change variables to q = p-p´.



The arguments of the δ functions are:



and we can neglect the q2 term in favor of the q term (since we assume low T which makes q small), and so the two δ’s would then be equivalent to each other. Then we can combine terms to obtain,



Pointing our q coordinate system’s z-axis in the direction of p, we have:



Filling in M,



Now consider an electron at the Fermi surface. Then we have εpF = μ, and so,



Now we can change variables,



Recalling the function definitions from stat mech file,



we can write this as:



But we have only found the (momentum) relaxation rate, not the transport scattering time. And so we simply insert the 1 – cos(**p**,**p**´) factor into the expression for 1/τ. Note for small momentum transfers:



where **q** = **p** – **p**´ of course, so that:



And now we could say,



And so we’d have:



Thus σ ~ 1/T5 for low T, or equivalently, ρ ~ T5. Makes sense that resistivity should vanish in small T limit. This is the Bloch-Gruneisen law. So altogether we have something like:

