**Thermal Equilibrium Properties**

We’ll take a look at a variety of thermal properties…well basically, I’ll just copy our work from the Statistical Mechanics folder/Ideal Fermionic gas. So we’ll approximate the energy spectrum as parabolic with a renormalized mass, m\*, which we’ll promptly call m again. For fermions, the grand partition function is:



The sum over states can be done as follows. For a cubic box of side length L, the allowed k-states would be, from QM, and imposing periodic boundary conditions for simplificity, **k** = 2π**n**/L. And so the volume occupied by an eigenstate is, in k-space, (2π)3/V. Accounting for spin up/down, we can say that the volume in k-space occupied by a state is 4π3/V. So we have:



It’s worthwhile to go to energy space too. In that case we can say,



So generally we have:



where the ρ’s are densities of states per unit volume. Taking a derivative w/r to μ and then going to the continuum limit, we have for the occupation number per energy.



We can make some simplifications:



This integral shows up a bit, so let’s bother to define:



which has the following asymptotic properties:



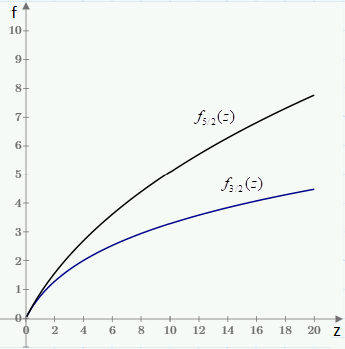
Also good to know that:



A useful differential property, for when we want to evaluate something like heat capacity, say:



We’ll recall that z ranges from 0 (high T) to ∞ (low T) for fermions (see Independent Particles file). Typical plots are:



So they start at 0 and logarithmically approach ∞.

**Chemical Potential**

We’ll want to get the chemical potential in terms of n. There is no general non-transcendental formula for it. It’ll suffice to examine the two asymptotic limits. First consider small T (large z) limit. Then out to second non-vanishing order:



which is:



To solve such equations we can use perturbation theory. Since lnz is large, 1/√lnz is small. So write equation as:



where λ is a small parameter. And let’s try to get a solution in a perturbative series in λ.



But we’ll only go out to O(λ), i.e., second non-vanishing order, because we only expanded f out to this order and so our solution will only be good to this order too. So plugging this ansatz into our equation, and expanding out to O(λ):



Equating both sides we have:



where we’ve recalled the definition of EF from somewhere. So we have:



and finally,



The expansion itself itself suggests it is good for kBT/EF small, i.e., kBT << EF. Or in other words, T << EF/kB = TF. Now EF ~ couple eV’s and so this corresponds to a temperature TF ~ 104K, well past the boiling point of most solids. So looks like we can use even just the first term at room temperature. Still, in the large T, small z limit, we have:



and so,



So, then to summarize:



**Internal Energy**

The energy could be obtained via thermodynamic relations, but easier to get it from our knowledge of n(ε). This is just:



Let’s expand this for large/small T. So for small T, large z, we have, going out to second order (and note we can only keep terms up to O(T2) because that’s as well as we know μ:



Filling in lnz expansion, and keeping only up to O(T2) we gots,



We can simplify the prefactor,



So we have:



The first term is just the familiar energy of the ground state of a Fermi gas. And the next term takes the temperature change into account. What about high T? Going back to the z expression and expanding for small z we have:



Filling in z for high T:



and this is the classical expectation. So altogether,

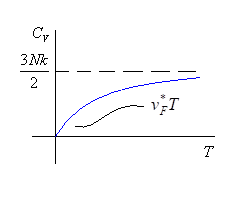


**Heat Capacity**

The derivative w/r to T would give us the heat capacity per unit volume cV = ∂u/∂T)V. Multiplying by V will give us the regular heat capacity. So,



where ρF = 3n/2EF = m\*kF/π2 (being explicit about the m\* thing now) is the density of states at the Fermi-energy. And overall, it looks something like:



So at high T it approaches the classical value. Note that it would only get there when T ~ TF ~ 104K. By then it would’ve melted of course. So at room temperatures, we’re still in the linear regime, and it’s so small as to make a negligible contribution to a typical solid’s heat capacity. This is why we don’t include mobile electrons in a metal’s heat capacity, in physics 1.

**Entropy**

Think I’ll get the entropy too, for later use, reprising the derivation in the independent particles file:



So,



and so,



Nifty formula that gets a lot of use. And one can get the heat capacity from *it*, using Cv = TdS/dT.

**Pressure**

Now let’s consider the pressure. Have to be careful here and not go to the continuum limit first, as this will wash out the ground state contribution, which is most important.



So we have in the limit,



The high T limit matches classical expectations. The low T limit is non-zero however and corresponds to our expectations of residual KE in the ground state.

**Compressibility**

Let’s calculate the T = 0 compressibility. This is:



So we have:



(this is also 3/5p, which we might compare to ideal gas, whose compressibility is 1/p, and so the Fermi gas is less compressible in general, which makes sense from the Pauli-Exclusion principle). From the compressibility we can get the speed of sound. In the Classical Mechanics folder/Navier Stokes file somewhere, we found the speed of sound in terms of the Bulk Modulus (B = 1/κ). And so this works out to (well, this is the *isothermal* sound velocity because we calculated the *isothermal* compressibility, when really we should be calculating the *isentropic* sound velocity, using the *isentropic* compressibility, but at T = 0, the difference is probably negligible?):



So we have:



vF is super-fast though, like 1% speed of light. So this sound wave is insanely fast!