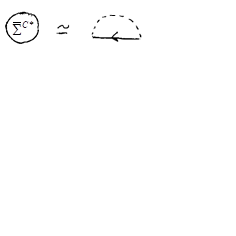
**Weak Disorder**

Let’s do a few examples…

**Self-Energy and Green’s function to first order using generic potential**

Let’s approximate the self-energy in the limits of low impurity concentrations and weak scattering potential. Then,



which is:



First, let’s demonstrate the connection to the energy shift and inverse lifetime we talked about above. Analytically continuing to the real frequency axis via iωn → ω + iη we have



Where have separated the self energy into its real and imaginary parts. Note the similarity to the BWPT formula – well it’s identical. Now we’re going to approximate the energy shift and decay rate (1/2 of it) by evaluating the self-energy ‘on-shell’ ω = ξk. We’ll continue with just the real part for now…



This result is what we expect from regular perturbation theory. We don’t see the 1st order shift in the energy because we are looking at the second order term (and because we subsumed the first order shift into the chemical potential – effectively setting it equal to zero). And for the imaginary part we have:



This is also what we expect from time dependent perturbation theory. But know that evaluating the self-energy won’t simply generate the single particle RSPT terms for a potential NiVi. Because we have to disorder average and take into account Fermi statistics. OK, back to C\*(k,iωn). We’ll use:



So that:



Note that it is the real part that contains the correction to the energy of the free electron state, and the imaginary part that contains the correction to the lifetime. Let us evaluate this for k close to the Fermi surface, which is the case of interest anyway, when we get to transport (because the thermal distribution function is differentiated, which results in a delta function near the Fermi surface). Now (noting that μ is roughly the Fermi energy) for k near kF and small q we can write.



So now we have,



So the integral for the energy shift is roughly odd about the Fermi surface. Of course as q increases, our linearization of the energy spectrum near the Fermi surface becomes less and less accurate. But even when you stray from the surface, the integral is suppressed by the denominator which goes as roughly q2, and the modulus squared potential as well, which, taking the screened Coulomb potential as an example, goes as 1/q4. The net denominator therefore goes as q6. This is pretty strongly damped, and so over the range for which the denominator doesn’t overwhelm the numerator, the numerator is approximately odd. Therefore we will get approximately zero for this integral. Or at least we can say that the result doesn’t depend much on k, so it will be an effective constant and we can just renormalize the chemical potential by subsuming this term in it. This basically amounts to saying that the energy shift is so small as to be insignificant. We will assume this to roughly be the case for all k. Certainly since only k near the fermi surface will come into play once we thermal average we can safely so generalize. So we can approximate the self energy as only having an imaginary part. Moving on to this then, we can write it in terms of the decay rate 1/τk defined above. Start with,



Now gonna introduce a delta function,



Well I think we restrict our k to vicinity of kF, first of all. And then we’ll say that the iωn/(ωn2 + ξk´´2) guy will damp the rest of the integrand so that only k´´ near kF, i.e., near k, will matter (because this is where ξk´´ = 0 roughly and iωn/(ωn2 + ξk´´2) is at a maximimum. This approximation might seem kind of tenuous, but I’ll note that our result, when analytically continued to get R(k,ω), and then subsequently evaluated on-shell R(k,ξk) gives what we got above, the known QM result for the scattering rate, to first order in V. So now we say,



So now our expression is decoupled. And we have:



so now we have, approximately:



which is what we’d expect (well except for the sgn function, though this bears resemblance to the causal GF’s sign switch at the fermi surface when T = 0) from second order time-dependent PT in Quantum Mechanics (but again, note that this correspondance won’t hold since the simple one particle QM formulas don’t account for particle statistics and disorder averaging). Note we shouldn’t interpret τk here as the time in between collision per se’, but rather the time it takes for the electron wave to lose its plane wave character, and to begin diffusive motion, i.e., sort of random motion. And if impurities are ‘weak’, then it may take several collisions for this to occur. Likewise, we shouldn’t take ℓ to be the distance in between collisions, but rather the distance over which it loses its plane wave character. If impurities are weak, then ℓ may be much larger than the average distance between impurities.Therefore we have that the temperature Green’s function is:



Note how it is analytic everywhere except at the boundary between the u.h.p. and l.h.p. This behavior is expected to persist. In the u.h.p., the ‘pole’ is in the l.h.p., and in the l.h.p., the ‘pole’ is in the u.h.p. So the pole is never in the region where the function is defined. To get the retarded/advanced GF’s, it’s best to manipulate GC\* into,



and then,



and so



And this ensures, note, that the retarded Green’s function is analytic in the u.h.p., while the advanced Green’s function is analytic in the l.h.p. Anyway, these results, we could of course have anticipated – knowing what the self energy means. But I wonder – I thought that the energy shift would have been the important contribution from the self energy since electrons near the Fermi surface shouldn’t have an appreciable scattering rate. In fact, aren’t electrons at the Fermi surface exact eigenstates (of sorts)? Well, no – that’s for e – e interactions, but for the impurity potential interaction, there is definitely an appreciable scattering rate!

If we want, we can now get GR in temporal space.



Squaring it evinces the decay rate. Thus we see that an electron prepared with momentum k, does indeed have a finite lifetime. The momentum relaxes according to the relaxation time τk-1. So we ought to expect that if we shot an electron with the specified momentum into a solid, its state would decay via impurity collisions according to our result. We can go on to calculate the retarded Green’s function in position space.



Where ρ(EF) is the density of states (per unit volume) at the Fermi level. And the position space Green’s function is a damped spherical wave. It is damped on the order of ℓ, which is the distance that it ought to travel until it loses its plane wave state. This is precisely what we would expect. Now we look at the spectral function:



Thus we see that the impurity potential has spread out the delta function spectral function for the free particle into a Lorentzian. Now let’s look at the density of states (per unit volume).



where in the 5th line I introduced the density of states, made the presence of the chemical potential explicit, and assumed the scattering rate was roughly constant per k. This is sort of the result we’d expect – recall in the QM folder our 1D delta potential for a periodic box only ‘marginally’ changed the k values and energies at which we had states. It added up to a decent change in GS energy in the end, but only because we have a macroscopic # of electrons. Can also say that we do not expect that the density of states will depend on position, since the disorder has been averaged over, restoring us to a translationally invariant condition.

**Self Energy and Green’s function to first order** **using δ potential**

To be concrete, we will use choose a particular form for the potential in our second order evaluation of the self energy.



This has the unfortunate consequence of making blow up – because the Fourier Transform doesn't go to 0, but is finite. This is physically saying that the energy shift due to a delta function potential is infinite? Well at least this indicates that our perturbative approach doesn’t work in this case. Anyway, we have…



(where we changed variables to k’ = k + q) So as we can see, the real part is divergent, regardless of the singularity, as the integral goes as ρ(ε)/ε ~ 1/√ε in 3D, which isn’t integral obviously. But we’ll just ignore this since as argued above, if we had a more realistic, i.e., damped V(q), we would find that the real part is exceedingly small compared to the imaginary part. And speaking of, the imaginary part gives,

,

Where ρ(x) is the free particle (that is with respect to the unperturbed system) density of states. Observe that when ω < -μ, ρ(ω+μ) = 0, and so τ-1(ω < -μ) = 0 too. Note how the lifetime of the state – roughly the uncertainty in its lifetime - is inversely proportional to the density of states, which gives the spread in energy about a single state, or sort of its energy uncertainty. Now let’s look at the retarded Green’s function. Using our work above, we can see that:



We wish to reconstruct the Green’s function in position space. So we take the inverse Fourier transform,



Since the integrand goes as trig/k the integral is finite. Since r is positive, we need only close the contour in the u.h.p., which is lucky for us since that’s where the pole is. Note however, that if we’re interested in ω < -μ, then τ-1(ω) = 0, and we’ll have to reinsert the +i0+ factor we’ve left out from the iωn → ω + i0+ replacement. In either case, we’ll have a pole in the u.h.p. and in the l.h.p. which will factor in the following manner.



Where  is in the u.h.p. Using the residue theorem, we find,



We can see that if the scattering time were infinite, we’d just have are usual spherical waves, so that if we planted a particle at point r, with energy, ω, it would propagate forward/outward in a spherical wave fashion, with momentum k. But reintroducing the scattering time, we can get an idea of the length scale over which the spherical wave will be damped. We note that it is the presence of the self energy’s non zero scattering rate which causes us to have an exponentially damped spherical wave. Let’s work out what k+is. We have to solve:



This approximation to the root is fairly easily determined graphically (analytically too). Just consider the ray pointing to the coordinates of k2 and then recognize that (for small imaginary part) the ray pointing at half the angle and square root the length is roughly square root real part (since it caries most of length) and ½ tan of the angle (as this is roughly ½ the angle itself). In any event, this simplifies to:



So we see that G goes is approximately equal to,



And we see that the Green’s function is spatially damped on the order of ℓ(ω) (we have to square the Green’s function).

*Recalculation with linearization of spectrum near the Fermi Surface approximation*

Now for practice we’ll return to our calculation via another approach. We’ll linearize the spectrum near the Fermi surface and recalculate our position – energy space Green’s function. Let



Then,



It is argued that the integral will come mostly from k ~ kF, and that for larger or smaller values of k, the integrand will wash out their contributions. Therefore, we can extend the range of integration to .



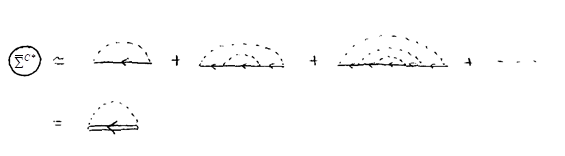
This obviously isn’t what we got before. But…for small ω << μ, this reduces to . And then,



Which is our previous result. So linearizing the spectrum is O.K. as long as we stick to energies, ω close to the Fermi level. So mathematical moral of the story is that its usually OK to approximate a polynomial in the denominator (which doesn’t have a zero along the path of integration) linearly about its maximum.

**Self-Consistent Born Approximation with δ potential**

So in the previous file we introduced the self-consistent Born approximation for the self-energy:



And of course this could be written out as:



with the solution,



And according to our self-consistent self-energy diagram above, we also have:



Now let’s work this out using the δ function potential,



Then we have:



So we have two equations and two unknowns. We can fill our expression for C\* into the self-energy equation, and get:



Might observe that if we neglect the Σ in the denominator, then we get our first order self-energy approximation from a previous example. We’ll note that we can change variables in the integral q + k → q´, which we’ll just designate as q again, which will give us:



But this means that C\* doesn’t depend on k since the RHS doesn’t. So now we can say,



and we can do this integral, kind of. First I guess we’ll change variables to ε,



where ρ(ε) is the (spin-less) density of states. But it looks like we’ll run into some integrability issues again. Let’s separate into real and imaginary parts,



Can see the real part will not converge, but we’ll argue like we did when analyzing the first order term, that if we had a more realistic, i.e., damped V(q), the real part would be insignificant, especially for k near the Fermi surface. But like before, the imaginary part does converge, so we’ll focus on that.



Going to go the cheap route and approximate the density of states as constant with the value equal to that where the integrand is largest, which would be at ε = μ, i.e., at the Fermi surface. So we’ll approximately have:



and so we come to,



which is, underwhelmingly, what we had before. So maybe we get zero for all diagrams higher than 2nd order when using a δ function potential (at least with our approximation of constant density of states over the entire range, and that we could extend μ´ to ∞ in that integral)?